A TEXT-BOOK

ON THE

MECHANICS OF MATERIALS

AND OF

BEAMS, COLUMNS, AND SHAFTS.

BY

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PREFACE.

The following pages contain an elementary course of study in the resistance of materials and the mechanics of beams, columns and shafts, designed for the use of classes in technical schools and colleges. It should be preceded by a good training in mathematics and theoretical mechanics, and be followed by a special study of the properties of different qualities of materials, and by detailed exercises in construction and design.

As the plan of the book is to deal mainly with the mechanics of the subject, extended tables of the results of tests on different kinds and qualities of materials are not given. The attempt, however, has been made to state average values of the quantities which express the strength and elasticity of what may be called the six principal materials. On account of the great variation of these values in different grades of the same material the wisdom of this attempt may perhaps be questioned, but the experience of the author in teaching the subject during the past seven years has indicated that the best results are attained by forming at first a definite nucleus in the mind of the student, around which may be later grouped the multitude of facts necessary in his own particular department of study and work.

As the aim of all education should be to develop the powers of the mind rather than impart to it mere information, the author has endeavored not only to logically set forth the principles and theory of the subject, but to so arrange the matter that students will be encouraged and required to think for themselves. The problems which follow each article will be found very useful for this purpose. Without the solution of many numerical exercises it is indeed scarcely possible to become well grounded in the theory.

In the chapters on flexure many problems relating to I beams and other wrought iron shapes are presented. The subject of continuous beams is not developed to its full extent, but it is thought that enough is given for an elementary course. The resistance of columns has been treated with as much fullness as now appears practicable from a theoretical point of view. Considerable attention has been paid to combined stresses, and particularly to the combination of torsion and flexure in shafts. A new formula for the case of repeated stresses is offered as a substitute for those of Launhardt and Weyrauch. The attempt has been made throughout to render the examples, exercises and problems of a practical nature, and also of a character to clearly illustrate the principles of the theory and the methods of investigation.

MANSFIELD MERRIMAN.

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In scientiis ediscendis prosunt exempla magis quàm præcepta.

NEWTON.

A TEXT-BOOK

ON THE

MECHANICS OF MATERIALS

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CHAPTER I.

ON THE RESISTANCE AND ELASTICITY OF MATERIALS.

ART. 1. AVERAGE WEIGHTS.

The principal materials used in engineering constructions are timber, brick, stone, cast iron, wrought iron and steel. The following table gives their average unit-weights and average specific gravities.

Material.	Average Weight.		Average Specific Gravity.
	Pounds per Cubic Foot.	Kilos per Cubic Meter.	
Timber	40	600	0.6
Brick	125	2 000	2.0
Stone	160	2 560	2.6
Cast Iron	450	7 200	7.2
Wrought Iron	480	7 700	7.7
Steel	490	7 800	7.8

These weights, being mean or average values, should be carefully memorized by the student as a basis for more precise knowledge, but it must be noted that they are subject to more or less variation according to the quality of the material. Brick, for instance, may weigh as low as 100, or as high as 150 pounds per cubic foot, according as it is soft or hard pressed. Unless otherwise stated the above average values will be used in the examples and problems of this book. In all engineering reference books are given tables showing the unit-weights for different qualities of the above six principal materials, and also for copper, lead, glass, cements and other materials used in construction.

For computing the weights of bars, beams and pieces of uniform cross-section, the following simple rules will often be found convenient.

A wrought iron bar one square inch in section and one yard long weighs ten pounds.
Steel is about two per cent heavier than wrought iron.
Cast iron is about six per cent lighter than wrought iron.
Stone is about one-third the weight of wrought iron.
Brick is about one-fourth the weight of wrought iron.
Timber is about one-twelfth the weight of wrought iron.

For example, consider a bar of wrought iron $1\frac{1}{2} \times 3$ inches and 12 feet long. Its cross-section is 4.5 square inches, hence its weight is $45 \times 4 = 180$ pounds. A steel bar of the same dimensions will weigh about 184 pounds, and a cast iron bar about 169 pounds.

Problem 1. How many square inches in the cross-section of a wrought iron railroad rail weighing 67 pounds per square yard? In a steel rail? In a wooden beam?

Prob. 2. Find the weights of a wooden beam $3 \times 4\frac{1}{2}$ inches in section and 13 feet long, and of a steel bar one inch in diameter and 13 feet long.

ART. 2. STRESSES AND STRAINS.

A 'stress' is a force which acts upon a body and tends to change its shape. If a weight of 400 pounds be suspended by a rope, the stress on the rope is 400 pounds. This stress produces an elongation of the rope which increases until the internal molecular forces or resistances are in equilibrium with the exterior stress. Stresses are measured in pounds, tons or kilograms. A 'unit-stress' is the amount of stress on a unit of area; this is expressed either in pounds per square inch, or in kilograms per square centimeter. Thus, if a rope of two square inches cross-section sustains a stress of 400 pounds, the unit-stress is 200 pounds per square inch, for the total stress must be regarded as distributed over the two square inches of cross-section.

A 'strain' is the amount of change of shape of a body caused by an applied stress. For instance, if a load be put on a pillar its length is shortened and the amount of shortening is a strain. So in case of the rope, the amount of elongation is a strain. Strains are generally measured in inches or centimeters. In popular language the word strain is usually synonymous with stress and indicates force, and is often so used in technical literature, but in the strict language of science it means the effect of the force in deforming the body. The measure of a stress is a weight, while that of a strain is the length of a line.

Three kinds of simple stress are produced by forces which tend to change the shape of a body. They are,

Tensile, tending to pull apart, as in a rope.
Compressive, tending to push together, as in a column.
Shearing, tending to cut across, as in punching a plate.

The nouns corresponding to these three adjectives are Tension, Compression, and Shear. The stresses which occur in beams, columns, and shafts are of a complex character, but they may always be resolved into the three kinds of simple stress. The first effect of a stress is to cause a deformation, or strain, in the body. This strain receives a special name according to the kind of stress which produces it. Thus,

Tension produces a tensile strain, or elongation.
Compression produces a compressive strain, or shortening.
Shear produces a shearing strain, or detrusion.

This change of shape is resisted by the forces between the molecules of the body, and as soon as this internal resistance balances the exterior stresses the change of shape ceases and the body is in equilibrium. But if the stresses be increased far enough the molecular resistances are finally overcome and the body breaks or ruptures.

Tension and Compression are similar in character but differ in regard to direction. A tensile stress upon a bar occurs when two forces of equal intensity act upon its ends, each in a direction away from the other. In compression the direction of the forces is reversed and each acts toward the bar. Evidently a simple tensile or compressive stress upon a bar is to be regarded as evenly distributed over the area of its cross-section, so that if P be the total stress in pounds and A the area of the cross-section in inches, the unit-stress is $\frac{P}{A}$ in pounds per square inch.

Shear requires the action of two forces exerted in parallel planes and very near together, like the forces in a pair of shears, from which analogy the name is derived. Here also the total shearing stress P is to be regarded as distributed uniformly over the area A, so that the unit-stress is $\frac{P}{A}$. And conversely if S represent the uniform unit-stress the total stress P is A times S.

In any case of simple stress acting on a bar let P be the total stress, A the area over which it is uniformly distributed, and S the unit-stress. Then,

$$(1) \qquad P = AS.$$

Also let λ be the total strain or deformation produced by the stress, l the length of the bar, and s the average strain per unit of length. Then also may be written,

$$(1)' \qquad \lambda = ls.$$

The laws implied in the statement of these two formulas are confirmed by experiment, if the stress be not too great.

Unit-stress in general will be denoted by S, whether it be tension, compression, or shear. S_t will denote tensile unit-stress, S_c compressive unit-stress, and S_s shearing unit-stress, when it is necessary to distinguish between them.

Prob. 3. A wrought iron rod 1½ inches in diameter breaks under a tension of 67 500 pounds. Find the breaking unit-stress.

Prob. 4. If a wooden bar 1½ × 3 inches breaks under a tensile stress of 30 000 pounds, what stress will break a bar 2 × 3½ inches?

ART. 3. EXPERIMENTAL LAWS.

Numerous tests or experiments have been made to ascertain the strength of materials and the laws that govern stresses and strains. The resistance of a rope, for instance, may be investigated by suspending it from one end and applying weights to the other. As the weights are added the rope will be seen to stretch or elongate, and the amount of this strain may be measured. When the load is made great enough the rope will break, and thus its ultimate tensile stress is known. For stone, iron, or steel, special machines, known as testing machines, have been constructed by which the effect of different stresses on different qualities and forms of materials may be accurately measured.

All experiments, and all experience, agree in establishing the five following laws, which may be regarded as the axioms of the science of the strength of materials.

(*A*)—When a small stress is applied to a body a small strain is produced, and on the removal of the stress the body springs back to its original form. For small stresses, then, materials may be regarded as perfectly elastic.

(*B*)—Under small stresses the strains, or changes of shape, are approximately directly proportional to the forces which produce them.

(*C*)—When the stress is great enough a strain is produced which is partly permanent, that is, the body does not spring back entirely to its original form on removal of the stress. The permanent part of the strain is termed a set. In such cases the strain is not proportional to the stress.

(*D*)—When the stress is greater still the strain rapidly increases and the body finally ruptures.

(*E*)—A sudden stress, or shock, is more injurious than a steady stress or than a stress gradually applied.

The words small and great, used in stating these laws, have, as will be seen later, very different values and limits for different kinds of materials and stresses. Let S be any unit-stress and s the unit-strain produced by it. Then according to the law (*B*) the ratio $\frac{S}{s}$ is a constant for small stresses, but its value for cast iron is about ten times its value for timber.

The 'ultimate strength' of a material under tension, compression, or shear, is the greatest unit-stress to which it can be subjected. This occurs at or shortly before rupture, and its value is very different for different materials.

Prob. 5. If a wrought iron bar 1 inch in diameter and 4 feet long elongates half an inch under a certain small stress P, how much will a bar 1¼ inches in diameter and 5 feet long elongate under a stress 2 P ?

Art. 4. Elastic Limit and Coefficient of Elasticity.

The 'elastic limit' is that unit-stress at which the permanent set is first visible and within which the stress is directly proportional to the strain. For stresses less than the elastic limit bodies are perfectly elastic, resuming their original form on removal of the stress. Beyond the elastic limit a permanent alteration of shape occurs, or, in other words, the elasticity of the material has been impaired. It is a fundamental rule in all engineering constructions that the material must never be strained beyond its elastic limit.

The 'coefficient of elasticity' of a bar for tension, compression, or shearing, is the ratio of the unit-stress to the unit-strain, provided the elastic limit of the material be not exceeded. Let S be the unit-stress, s the unit-strain and E the coefficient of elasticity. Then by the definition,

$$(2) \qquad E = \frac{S}{s}.$$

By law (B) the quantity E is a constant for each material, until S reaches the elastic limit. Beyond this limit s increases more rapidly than S and the ratio is no longer constant. Equation (2) is a fundamental one in the science of the strength of materials. Since E varies inversely with s, the coefficient of elasticity may be regarded as a measure of the stiffness of the material. The stiffer the material the less is the change in length under a given stress and the greater is E. The values of E for materials have been determined by experiments with testing machines and their average values will be given in the following articles. E is necessarily expressed in the same unit as the unit-stress S.

Another definition of the coefficient of elasticity is that it is the unit-stress which would elongate a bar to double its original length, provided that it could be done without exceeding the elastic limit. That this is in agreement with (2) may be shown by regarding a bar of length l which elongates the amount λ under the unit-stress $\frac{P}{A}$. Then (2) becomes,

$$E = \frac{P}{A} \div \frac{\lambda}{l} = \frac{Pl}{A\lambda},$$

and if λ be equal to l, E is the same as the unit stress $\frac{P}{A}$.

Prob. 6. Find the coefficient of elasticity of a bar of wrought iron 1¼ inches in diameter and 16 feet long which elongates ⅛ inch under a tensile stress of 15 000 pounds.

Prob. 7. If the coefficient of elasticity of cast iron is 15 000 000 pounds per square inch, how much will a bar 2 × 3 inches and 6 feet long stretch under a tension of 5 000 pounds?

Ans. 0.004 inches.

Art. 5. Tension.

The phenomena of tension observed when a gradually increasing stress is applied to a bar, are briefly as follows: When the unit-stress S is less than the elastic limit S_e, the unit-elongation s is small and proportional to S. Within this limit the ratio of S to s is the coefficient of elasticity of the material. After passing the elastic limit the bar rapidly elongates and this is accompanied by a reduction in area of its cross-section. Finally when S reaches the ultimate tensile strength S_t, the bar tears apart. Usually S_t is the maximum unit-stress on the bar, but in some cases the unit-stress reaches a maximum shortly before rupture occurs.

The constants of tension for timber, cast iron, wrought iron and steel are given in the following table. The values are average ones and are liable to great variations for different grades and qualities of materials. Brick and stone are not here mentioned, as they are rarely or never used in tension.

Material.	Coefficient of Elasticity, E.	Elastic Limit, S_e.	Ultimate Tensile Str'gth, S_t.	Ultimate Elongation, s.
	Lbs per sq. in.	Lbs per sq. in.	Lbs per sq. in.	In. per linear in.
Timber	1 500 000	3 000	10 000	0.015
Cast Iron	15 000 000	6 000	20 000	0.005
Wrought Iron	25 000 000	25 000	55 000	0.15
Steel	30 000 000	40 000	100 000	0.10

The values of the coefficients of elasticity, elastic limits and breaking or ultimate strengths are given in pounds per square inch of the original cross-section of the bar. The ultimate elongations are in fractional parts of the original length, or they are the elongations per linear unit; these elongations, should be regarded only as very rough averages, since they are subject to great variations depending on the shape, size and quality of the specimen.

The ultimate elongation, together with the reduction in area of the cross-section, furnishes the means of judging of the ductility of the material. The reduction of area in cast iron and in many varieties of steel is scarcely perceptible, while in other varieties of steel and in wrought iron it may be as high as 0.4 of the original section.

A graphical illustration of the principal phenomena of tension is given in Fig. 1. The unit-stresses are taken as ordinates and

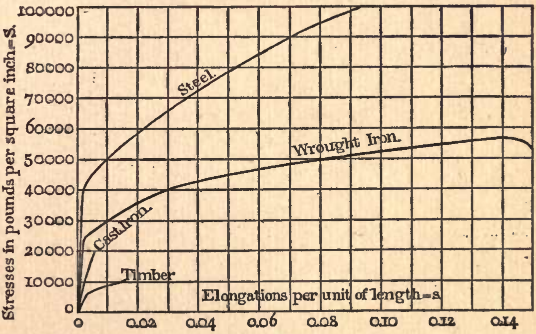


Fig. 1.

the unit-elongations as abscissas. For each unit-stress the corresponding unit-elongations as found by experiment are laid off and curves drawn through the points thus determined. For each of the materials the curve is a straight line from the origin until the elastic limit is reached, as should be the case according to the law (B). The tangent of the angle which this line makes with the axis of abscissas has evidently the same value as the coefficient of elasticity of the material. At the elastic limit a sudden change in the curve is noticed and the elongation rapidly increases. The termination of the curve indicates the point of rupture. These curves show more plainly to the eye than the values in the table can do the differences in the properties of the materials. It will be seen that the elastic limit is not a well defined point, but that its value is more or less uncertain, particularly for cast iron and timber. It should be also clearly understood that particular curves for special cases would often show great variations from their mean forms as represented in the diagram.

As a particular example a tensile test of a wrought iron bar ¾ inches in diameter and 12 inches long made at the Pencoyd Iron Works will be considered. In the first column of the following table are given the total stresses which were successively applied, in the second the stresses per square inch, in the third the total elongations, and in the fourth the elongations or sets after removal of the stress. The unit-elongations are found by dividing those in the table by 12 inches, the length of the specimen. Then from formula (2) the coefficient of elasticity can be computed for different values of S and s. Thus for the second, third and sixth cases,

Total Stress. in Pounds.	Stress per Square Inch.	Elongation.	
		Load on.	Load off.
2 245	5 000	.001	.000
4 490	10 000	.004	.000
6 735	15 000	.005	.000
8 980	20 000	.008	.000
9 878	22 000	.009	.000
10 776	24 000	.010	.000
11 674	26 000	.0105	.000
12 572	28 000	.011	.000
13 470	30 000	.013	.000
14 368	32 000	.014	.000
15 266	34 000	.015	.002
16 164	36 000	.022	.007
17 062	38 000	.416	.3995
17 960	40 000	.5445	.523
25 450	50 000	1.740	1.707
23 175	51 600	2.468	......

Specimen broke with 51 600 lbs. per square inch.
Stretch in 12 inches 2.468 inch.
" 8 " 1.812 "
" 8 " 22.65 per cent.
Fractured area 0.297 square inches.

$$\text{for } S = 15\,000,\ s = \frac{0.005}{12} \text{ and } E = 36\,000\,000$$

$$\text{for } S = 20\,000,\ s = \frac{0.008}{12} \text{ and } E = 30\,000\,000$$

$$\text{for } S = 26\,000,\ s = \frac{0.0105}{12} \text{ and } E = 29\,700\,000$$

The elastic limit was reached at about 33 000 pounds per square inch, indicated by the beginning of the set and the rapid increase of the elongations. The ultimate tensile strength of the specimen was 51 600 pounds per square inch. The ultimate unit-elongation was $\frac{2.468}{12} = 0.205$ inches per linear inch. It hence appears that this bar of wrought iron was much higher than the average as regards stiffness, elastic limit and ductility, and lower than the average in ultimate strength.

The 'working strength' of a material is that unit-stress to which it may safely be subjected. This should never be greater than the elastic limit of the material, since if that limit be exceeded there is a permanent set which impairs the elasticity. In order to secure an ample margin of safety it is customary to take the working strength at from one-third to two-thirds the elastic limit S_e. The reasons which govern the selection of exact values of the working strength will be set forth in the following articles.

To investigate the security of a piece subjected to a tension P, it is necessary first to divide P by the area of the cross-section and thus determine the working strength. Then a comparison of this value with the value S_t for the given material will indicate whether the applied stress is too great or whether the piece has a margin of safety. For example, if a tensile stress of 4 500 pounds be applied to a wrought iron bar of ¾ inches diameter the working unit-stress is,

$$S = \frac{P}{A} = \frac{4\,500}{0.449} = 10\,000 \text{ pounds per square inch, nearly.}$$

As this is less than one-half the elastic limit of wrought iron the bar has a good margin of security.

To design a piece to carry a given tension P it is necessary to assume the kind of material to be used and its allowable working strength S. Then $\frac{P}{S}$ is the area of the cross-section of the piece, which may be made of such shape as the circumstances of the case require. For example, if it be required to design a wooden bar to carry a tensile stress of 4 500 pounds, the working strength may be assumed at 1 000 pounds per square inch and the required area is 4.5 square inches, so that the bar may be made 2 × 2¼ inches in section.

The elongation of a bar within the elastic limit may readily be computed by the help of formula (2). For instance, let it be required to find the elongation of a wooden bar 3 × 3 inches and 12 feet long under a tensile stress of 9 000 pounds. From the formulas (2) and (1),

$$E = \frac{S}{s} = \frac{P}{A} \div \frac{\lambda}{l} \qquad \therefore \lambda = \frac{Pl}{AE}$$

Substituting in this the values $E = 1\,500\,000$, $A = 9$, $l = 144$ and $P = 9\,000$, the probable value of the elongation λ is found to be 0.096 inches.

Prob. 8. Investigate the security of a cast iron bar 2 × 2 inches when subject to a tension of 40 000 pounds.

Prob. 9. Find the size of a round wrought iron rod to safely carry a tensile stress of 100 000 pounds.

Prob. 10. Compute the elongation of wooden and of a cast iron bar, each being 2 × 3 inches and 16 feet long, under a tensile stress of 6 000 pounds.

Art. 6. Compression.

The phenomena of compression are similar to those of tension, provided that the length of the specimen does not exceed about five times its least diameter. The piece at first shortens proportionally to the applied stress, but after the elastic limit is passed the shortening increases more rapidly and is accompanied by a slight enlargement of the cross-section. When the stress reaches the ultimate strength of the material the specimen cracks and ruptures. If the length of the piece exceeds about ten times its least diameter a sidewise bending or flexure of the specimen occurs, so that it fails under different circumstances than those of direct compression. All the values given in this article refer to specimens whose lengths do not exceed about five times their least diameter. Longer pieces will be discussed in chapter V under the head of 'columns.' Owing to the difficulty of making experiments on short specimens and to an increase of resistance that arises with the enlargement of the cross-section, the phenomena of compression are not usually so regular as those of tension.

The constants of compression for short specimens are given in the following table, the values, like those for tension, being rough average values liable to much variation in particular cases.

Material.	Coefficient of Elasticity, E.	Elastic Limit, S_e.	Ultimate Compressive Strength, S_c.
	Lbs per sq. in.	Lbs per sq. in.	Lbs per sq. in.
Timber	1 500 000	3 000	8 000
Brick			2 500
Stone	6 000 000		6 000
Cast Iron	15 000 000		90 000
Wrought Iron	25 000 000	25 000	55 000
Steel	30 000 000	40 000	150 000

The values of the coefficient of elasticity and the elastic limit for timber, wrought iron and steel here stated are the same as those for tension, but the same reliance cannot be placed upon them, owing to the irregularity of experiments thus far made. There is reason to believe that both the elastic limit and the coefficient of elasticity for compression are somewhat greater than for tension.

The investigation of a piece subjected to compression, or the design of a short piece to be subjected to compression, is effected by exactly the same methods as for tension. Indeed it is customary to employ these methods for cases where the length of the piece is as great as ten times its least diameter.

Prob. 11. Find the height of a brick tower which crushes under its own weight. Also the height of a stone tower.

Prob. 12. The piston of a steam engine is 18″ in diameter and the piston rod is 2″. Find the compressive unit-stress on the piston rod when the steam pressure behind the piston is 80 pounds per square inch.

Prob. 13. Compute the amount of shortening in a wrought iron specimen 1 inch in diameter and 5 inches long under a load of 6 000 pounds.

ART. 7. SHEAR.

Shearing stresses and strains occur whenever two forces, acting like a pair of shears, tend to cut a body between them. When a plate is punched the ultimate shearing strength of the material must be overcome over the surface punched. When a bolt is in tension the applied stress tends to shear off the head and also to strip or shear the threads in the nut and screw. When a rivet connects two plates which transmit tension the plates tend to shear the rivet across.

The ultimate shearing strength of materials is easily determined by causing rupture under a stress P, and then dividing P by the area A of the shorn surface. The value of this for timber is found to be very much smaller along the grain than across the grain; for the first direction it is sometimes called longitudinal shearing strength and for the second transverse shearing strength. The same distinction is sometimes made in rolled wrought iron plates and bars where the process of manufacture induces a more or less fibrous structure. The elastic limit and the amount of detrusion for shearing are difficult to determine experimentally. The coefficient of elasticity however has been deduced by means of certain calculations and experiments on the twisting of shafts, explained in chapter VI under the head of torsion.

Material.	Coefficient of Elasticity, E.	Ultimate Shearing Strength, S_s.
Timber, Longitudinal	400 000	600
Timber, Transverse		3 000
Cast Iron	6 000 000	20 000
Wrought Iron	15 000 000	50 000
Steel		70 000

The investigation and design of a piece to withstand shearing stress is made by the means of the equation $P = AS$, in the same manner as for tension and compression. For instance, consider the cylindrical wooden specimen shown in Fig. 2, which has the following dimensions: length $ab = 6$ inches,

Fig. 2.

diameter of ends $= 4$ inches, diameter of central part $= 2$ inches. Let this specimen be subjected to a tensile stress in the direction of its length. This not only tends to tear it apart by tension, but also to shear off the ends on a surface whose length is ab and whose diameter is that of the central cylinder. The force P required to cause this longitudinal shearing is,

$$P = AS_s = 3.14 \times 2 \times 6 \times 600 = 22\ 600 \text{ pounds}.$$

while the force required to rupture the specimen by tension is,

$$P = AS_t = 3.14 \times 1^2 \times 10\ 000 = 31\ 400 \text{ pounds}.$$

As the former resistance is only about two-thirds that of the latter the specimen will evidently fail by the shearing off of the ends.

Prob. 14. A hole ¾ inches in diameter is punched in a wrought iron plate ⅝ inches thick by a pressure on the punch of 78 000 pounds. What is the ultimate shearing strength of the iron?

Prob. 15. A wrought iron bolt 1½ inches in diameter has a head ¾ inches long. Find the unit-stress tending to shear off the head when a tension of 3 000 pounds is applied to the bolt?

Art. 8. Factors of Safety and Working Stresses.

The factor of safety for a body under stress is the ratio of its ultimate strength to the actual existing unit-stress. The factor of safety for a piece to be designed is the ratio of the ultimate strength to the proper allowable working strength. Thus if S_t be the ultimate and S the working strength, the factor of safety f is $f = \frac{S_t}{S}$. The factor of safety is then always an abstract number, which indicates the number of times the working stress may be multiplied before the rupture of the body.

The law (E) in Art. 3 indicates that working stresses should be lower for shocks and sudden strains than for steady loads and varying stresses. In a building the stresses on the walls are steady, so that the working strength may be taken high and hence the factor of safety low. In a bridge the stresses in the several members are more or less varying in character which requires a lower working strength and hence a higher factor of safety. In a machine subject to shocks the working strength should be lower still and the factor of safety very high.

Twice as much strain is theoretically caused by a suddenly applied stress as by one gradually applied. The complete demonstration of this proposition belongs to the subject of dynamics and is of a too complex nature to be here given. It is however plain that when the load or stress is gradually applied that it uniformly increases from 0 to P so that it has an average value of $\frac{1}{2}P$ and performs the work $\frac{1}{2}P\lambda$ in causing a given elongation λ, while the same work would be performed in the same distance by the constant force $\frac{1}{2} P$. Hence it might be supposed that the sudden stress P would produce double the strain of the gradual stress whose average value is $\frac{1}{2} P$. Accordingly not to impair the material by inducing a set the working unit-stresses should be low for bodies subject to shocks.

The following are average values of the allowable factors of safety commonly employed in American practice. These values

Material.	For Steady Stress. (Buildings.)	For Varying Stress. (Bridges.)	For Shocks. (Machines.)
Timber	8	10	15
Brick & Stone	15	25	30
Cast Iron	6	10	15
Wrought Iron	4	6	10
Steel	5	7	10

are subject to considerable variation in particular instances, not only on account of the different qualities and grades of the material but also on account of the varying judgment of designers. They will also vary with the range of varying stress so that different parts of a bridge may have very different factors of safety.

The proper allowable working strength of any material for tension, compression or shearing, may be at once found by dividing the ultimate strength by the proper factor of safety. Regard should also be paid to the elastic limit in selecting the working strength, particularly for materials whose elastic limit is well defined. For wrought iron and steel the working strength should be well within the elastic limit, as already indicated in previous articles. For cast iron, stone, brick and timber it is often difficult to determine the elastic limit, and experience alone can guide the proper selection of the working strength. The above factors of safety indicate indeed the conclusions of experiment and experience extending over the past hundred years.

The student should clearly understand that the exact values given in this and the preceding articles would not be arbitrarily used in any particular case of design. For instance, if a given lot of wrought iron is to be used in an engineering structure, specimens of it should be tested to determine its coefficient of elasticity, elastic limit, ultimate strength and precentage of elongation. Then the engineer will decide upon the proper working strength, being governed by its qualities as shown by the tests, the character of the stresses that come upon it and the cost of workmanship.

The two fundamental principles of engineering design are stability and economy, or in other words :

First, the structure must safely withstand all the stresses which are to be applied to it.

Second, the structure must be built and maintained at the lowest possible cost.

The second of these fundamental principles requires that all parts of the structure should be of equal strength, like the celebrated "one-hoss shay" of the poet. For, if one part is stronger than another it has an excess of material which might have been spared. Of course this rule is to be violated if the cost of the labor required to save the material be greater than that of the material itself.

The factors of safety stated above are supposed to be so arranged that if different materials be united the stability of all parts of the structure will be the same, so that if rupture occurs, everything would break at once. Or, in other words, timber with a factor of safety 8 has about the same reliability as wrought iron with a factor of 4 or stone with a factor of 15.

The assignment of working strengths with regard to the elastic limits of materials is more rational than that by means of the factors of safety, and in time it may become the more important and valuable method. But at present the ultimate strengths are so much better known and so much more definitely determinable than the elastic limits that the empirical method of factors of safety seems the more important, due regard being paid to considerations of stiffness, elastic limit and ductility.

As an example, let it be required to find the proper size of a wrought iron rod to carry a steady tensile stress of 90 000 pounds. In the absence of knowledge regarding the quality of the wrought iron the ultimate strength S_t is to be taken as the average value, 55 000 pounds per square inch. Then, for a factor of safety of 4, the working strength is,

$$S = \frac{55\,000}{4} = 13\,750 \text{ pounds per square inch.}$$

The area of cross-section required is hence,

$$A = \frac{90\,000}{13\,750} = 6.6 \text{ square inches,}$$

which may be supplied by a rod of $2\frac{15}{16}$ inches diameter.

Prob. 16. Find the diameter in centimeters of a wrought iron rod to safely carry a steady stress of 20 000 kilograms. (See tables in the Appendix.)

Prob. 17. A wooden frame ABC forming an equilateral triangle consists of pieces 2 × 2 inches jointed at A, B and C. It is placed in a vertical plane and supported at B and C so that BC is horizontal. Find the unit-stress and factors of safety in each of the three pieces when a load of 4 000 pounds is applied at A.

Prob. 18. Determine the size of a short steel piston rod when the piston is 15 inches in diameter and the steam pressure 120 pounds per square inch.

CHAPTER II.

ON PIPES, CYLINDERS, AND RIVETED JOINTS.

ART. 9. WATER AND STEAM PIPES.

The pressure of water or steam in a pipe is exerted in every direction and tends to tear the pipe apart longitudinally. This is resisted by the internal tensile stresses of the material. If p be the pressure per square inch of the water or steam, d the diameter of the pipe and l its length, the force P which tends to cause longitudinal rupture is $p.ld$. This is evident from the fundamental principle of hydrostatics that the pressure of water in any direction is equal to the pressure on a plane perpendicular to that direction, or may be seen by imagining the pipe to be filled with a solid substance on one side of the diameter which would receive the pressure p on each square inch of the area ld and transmit it into the pipe. If t be the thickness of the pipe and S the working tensile strength of the material, the resistance on each side is $tl.S$. As the resistance must equal the pressure,

$$pld = 2tlS, \qquad \text{or} \quad pd = 2tS,$$

which is the formula for discussing pipes under internal pressure.

The unit-pressure p for water may be computed from a given head h by finding the weight of a column of water one inch square and h inches high. Or if h be given in feet, the pressure in pounds per square inch may be computed from $p = 0.434h$.

Water pipes may be made of cast or wrought iron, the former being more common, while for steam the latter is preferable.

Wrought iron pipes are sometimes made of plates riveted together but the discussion of these is reserved for another article. A water pipe subjected to the shock of water ram needs a high factor of safety, and in a steam pipe the factors should also be high owing to shocks liable to occur from condensation and expansion of the steam. The formula above deduced shows that the thickness of a pipe must increase directly as its diameter, the internal pressure being constant.

For example, let it be required to find the factor of safety for a cast iron water pipe of 12 inches diameter and ⅝ inches thickness under a head of 300 feet. Here p, the pressure per square inch, equals 130.2 pounds. Then from the formula the unit-stress is,

$$S = \frac{pd}{2t} = \frac{130 \times 12}{2 \times \frac{5}{8}} = 1\,230 \text{ pounds per square inch,}$$

and hence the factor of safety is,

$$f = \frac{20\,000}{1230} = \text{about } 16,$$

which indicates ample security against the shock of water ram.

Again let it be required to find the proper thickness for a wrought iron steam pipe of 18 inches diameter to resist a pressure of 120 pounds per square inch. With a factor of safety of 10 the working strength S is about 5 500 pounds per square inch. Then from the formula,

$$t = \frac{pd}{2S} = \frac{120 \times 18}{2 \times 5500} = 0.2 \text{ inches.}$$

In order to safely resist the stresses and shocks liable to occur in handling the pipes, the thickness is often made somewhat greater than the formula requires.

Prob. 19. What should be the thickness of a cast iron pipe of 18 inches diameter under a head of 300 feet?

Prob. 20. A wrought iron pipe is 4.5 inches in internal diameter and weighs 12.5 pounds per linear foot. What steam pressure can it carry with a factor of safety of 8?

Prob. 21. What head of water will burst a cast iron pipe of 24 inches diameter and ¾ inches thickness?

Art. 10. Cylinders and Spheres.

A cylinder subject to the internal pressure of water or steam tends to fail longitudinally exactly like a pipe. The head of the cylinder however undergoes a pressure which tends to separate it from the walls. If d be the diameter of the cylinder and p the internal pressure per square unit, the total pressure on the head is $\frac{1}{4}\pi d^2.p$. If S be the working unit-stress and t the thickness of the cylinder, the resistance to the pressure is $\pi dt\,S$. Since the resistance must equal the pressure,

$$\tfrac{1}{4}\pi d^2.p = \pi dt.S, \qquad \text{or} \quad pd = 4tS.$$

By comparing this with the formula of the last article it is seen that the resistance of a pipe to tranverse rupture is double the resistance to longitudinal rupture.

A sphere subject to internal pressure tends to rupture around a great circle, and it is easy to see that the conditions are exactly the same as for the transverse rupture of a cylinder, or that $pd = 4tS$. For very thick spheres and cylinders the formulas of this and the last article are only approximate.

A cylinder under external pressure is theoretically in a similar condition to one under internal pressure as long as it remains a true circle in cross-section. A uniform internal pressure tends to preserve and maintain the circular form of the cylindrical annulus, but an external pressure tends at once to increase the slightest variation from the circle and render it elliptical. The distortion when once begun rapidly increases and failure occurs by the collapsing of the tube rather than by the crushing of the material. The flues of a steam boiler are the most common instance of cylinders subjected to external pressure. In the absence of a rational method of investigating such cases recourse has been had to experiment. Tubes of various diameters, lengths, and thicknesses have been subjected to external pressure until they collapse and the results have been compared and discussed. The following for instance are the results of three experiments by Fairbairn on wrought iron tubes.

Length in Inches.	Diameter in Inches.	Thickness in Inches.	Pressure per Sq. Inch.
37	9	0.14	378
60	14½	0,125	125
61	18¾	0.25	420

From these and other similar experiments it has been concluded that the collapsing pressure varies directly as some power of the thickness, and inversely as the length and diameter of the table. For wrought iron tubes Wood gives the empirical formula for the collapsing pressure per square inch,

$$p = 9\,600\,000 \frac{t^{2.18}}{ld}$$

The values of p computed from this formula for the above three experiments are 397, 120 and 409, which agree well with the observed values.

The proper thickness of a wrought iron tube to resist external pressure may be readily found from this formula after assuming a suitable factor of safety. For example, let it be required to find t when $p = 120$ pounds per square inch, $l = 72$ inches, $d = 4$ inches and the factor of safety $= 10$. Then

$$t^{2.18} = \frac{10 \times 120 \times 72 \times 4}{9\,600\,000} = 0.036,$$

from which with the help of logarithms the value of t is found to be 0.22 inches.

Prop. 22. What internal pressure per square inch will burst a cast iron sphere of 24 inches diameter and ¾ inches thickness.

Prob. 23. What external pressure per square inch will collapse a wrought iron tube 96 inches long, 3 inches diameter and 0.25 inches thickness?

ART. 11. RIVETED JOINTS.

The strength of riveted joints which are subject to tension depends upon the shearing strength of the rivets and the tensile strength of the plates. Whatever be the arrangement all parts of the joint should have the same degree of stability, so that at the point of rupture both rivets and plates (like the one-hoss shay) may simultaneously fail.

A joint subject to tension is always weaker than the parts which it connects, since a portion of the material is removed to make room for the rivets. It may be required to arrange a joint so as to secure either strength or tightness. For a bridge, strength is mainly needed; for a gas holder, tightness is the principal requisite; while for a boiler both these qualities are desirable.

Case I. Lap Joint with single riveting.—Let P be the tensile stress which comes on one rivet, d the diameter of a rivet, t the thickness of the plates, and a the pitch of the rivets. Let S_t be the unit working strength of the plate and S_s that of the rivet, the former being in tension and the latter in shear. Then for the plate $P = t(a-d)\,S_t$ and for the rivet $P = \frac{\pi d^2}{4}\,S_s$. For equal strength these must be equal, or,

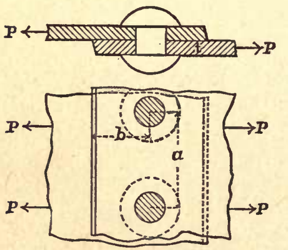


Fig. 3.

$$t\,(a-d)\,S_t = \frac{\pi d^2}{4}\,S_s.$$

For wrought iron plates and rivets S_t is about equal to S_s, and therefore,

$$a = d + \frac{\pi}{4} \cdot \frac{d^2}{t},$$

is the formula for finding the pitch in wrought iron lap riveting. By punching the hole the section of the plate is reduced from $a.t$ to $(a-d)\,t$, so that the ratio of the strength of the joint to that of the unriveted plate is,

$$r = \frac{a-d}{a} = \frac{1}{1 + \frac{4}{\pi}\,\frac{t}{d}}$$

This shows clearly that for a given thickness t, large rivets give the largest value of r, while small rivets give small values of r. The smaller the rivet the smaller the pitch and the greater the loss in strength. For example,

if $d = t$, $a = 1.78t$ and $r = 0.44$,

if $d = 3t$, a $10.1t$ and $r = 0.77$,

Hence when strength is required large rivets should be used, while to give tight joints small rivets must be used with a sacrifice of strength.

Case II. Lap Joint with double riveting.—In this arrangement twice as many rivets are used and hence

$$P = t(a-d)\,S_t = 2\,\frac{\pi d^2}{4} S_s$$

from which, for wrought iron,

$$a = \frac{\pi d^2}{2t} + d \text{ and } r = \frac{1}{1 + \frac{2t}{\pi d}}$$

For this case the same truth holds regarding strength, thus,

when $d = t$, $a = 2.6t$ and $r = 0.61$

when $d = 3t$, $a = 17.3t$ and $r = 0.83$

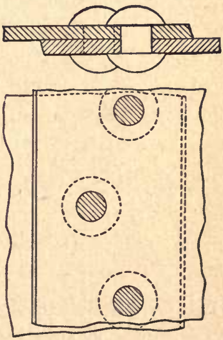


Fig. 4.

and the loss of strength is much less than in single riveting.

Case III. Butt Joint with single riveting.—For this arrangement the shear on the rivets comes on two cross-sections, and the covers need to be only one-half as thick as the plates. It is easy to deduce for wrought iron,

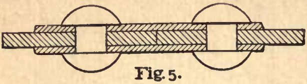


Fig. 5.

$$a = d + \frac{\pi d^2}{2t} \qquad \text{and} \qquad r = \frac{1}{1 + \frac{2t}{\pi d}}$$

which is the same as in the preceding case.

Case IV. Butt Joints with double riveting.—Here there are two rows of rivets and each is in double shear. It is easy to find for the case of wrought iron,

$$a = d + \frac{\pi' d^2}{t} \quad \text{and} \quad r = \frac{1}{1 + \frac{t}{\pi d}}$$

which indicates a large increase in strength over single riveting.

Compression is brought sidewise upon the rivets in all the above cases by means of the stress P and tends to cause failure by crushing. The exact manner in which the compression acts upon the cylindrical surface of the rivet is not known, but it is usually supposed to be equivalent to a stress uniformly distributed over the projection of the surface on a plane through the axis of the rivet. Thus for single riveting with either lap or butt joints,

$$P = tdS_c,$$

and for double riveting,

$$P = 2tdS_c.$$

This compressive or bearing resistance of rivets always needs to be regarded in actual cases of design. The lap of the plates is determined by practical considerations rather than by theoretic formulas.

For example, let it be required to investigate a single riveted butt joint consisting of plates 6 inches wide and 0.5 inches thick with covers 0.25 inches thick and 2 rivets of 1 inch diameter and 4 inches pitch, when under a tension of 16 000 pounds. First, the tensile unit-stress on the plate is,

$$S_t = \frac{16\ 000}{4 \times 0.5} = 8\ 000 \text{ pounds per square inch.}$$

Next the shearing unit-stress on the rivets is,

$$S_s = \frac{16\ 000}{2 \times 0.785} = 10\ 200 \text{ pounds per square inch.}$$

Lastly, the bearing compressive unit-stress on the rivets is,

$$S_c = \frac{16\ 000}{2 \times 1 \times 0.5} = 16\ 000 \text{ pounds per square inch.}$$

It hence appears that the joint has the greatest resistance against tension and the least against compression.

Prob. 24. A butt joint with double riveting has plates half an inch thick and rivets one inch in diameter. Find the pitch of the rivets and the percentage of strength lost by the joint.

Prob. 25. A boiler is to be formed of wrought iron plates ⅜ inches thick united by single lap joints with rivets ⅝ inches in diameter. Find the proper pitch of the rivets. Find the factor of safety of the boiler if it is 30 inches in diameter and carries a steam pressure of 100 pounds per square inch above the atmosphere.

ART. 12. MISCELLANEOUS EXERCISES.

It will be profitable to the student to thoroughly perform the following exercises and to write upon each a detailed report which should contain all the sketches and computations necessary to clearly explain the data, the reasoning, and the conclusions.

Exercise 1. Visit an establishment where tensile tests are made. Ascertain the kind of machine employed, its capacity, the method of applying the stresses, the method of measuring the stresses, the method of measuring the elongations. Ascertain the kind of material tested, the reason for testing it, and the conclusions derived from the tests. Give full data for the tests on four different specimens, compute the values of coefficient of elasticity, ultimate strength and ultimate elongation for them, and state your conclusions.

Exercise 2. Procure a wrought iron bolt and nut. Determine diameter of bolt, length of head, and length of nut. State the equation of condition that the head of the nut shall shear off at the same time the bolt ruptures under tension. Take the shearing strength as half the tensile and compute the length of head for a given diameter. State why theoretically the length of the nut should be double that of the head. Compare theory with practice.

Exercise 3. Go to a boiler shop and witness operations upon a boiler in process of construction. Ascertain length and diameter of boiler, thickness, pitch and diameter of rivets, method of forming holes, method of doing the riveting. Compute the loss of strength caused by the riveting. Compute the steam pressure which would cause longitudinal rupture of the plate along a line of rivets. Ascertain whether the joint is proportioned in accordance with theory.

Prob. 26. A bar whose cross-section is A is subjected to a tensile stress P. Prove that a shear exists along any oblique section and that the maximum shearing unit-stress is $\frac{1}{2}\frac{P}{A}$.

Prob. 27. A wrought iron pipe $\frac{3}{8}$ inches thick and 20 inches in diameter is to be subjected to a head of water of 230 feet. Compute the probable increase in diameter due to the internal pressure.

CHAPTER III.

ON SIMPLE BEAMS AND CANTILEVERS.

ART. 13. DEFINITIONS.

Transverse stress, or flexure, occurs when a bar is laid in a horizontal position upon one or more supports. The weight of the bar and the loads upon it cause it to bend and induce in it stresses and strains of a complex nature which, as will be seen later, may be resolved into those of tension, compression, and shear. Such a bar is called a beam.

A simple beam is a bar resting upon supports at its ends. A cantilever beam is a bar on one support in its middle, or the portion of any beam projecting out of a wall or beyond a support may be called a cantilever. A continuous beam is a bar resting upon more than two supports. In this book the word beam, when used without qualification, includes all kinds, whatever be the number of the supports or whether the ends be free, supported, or fixed.

The elastic curve is the curve formed by a beam as it deflects downward under the action of its own weight and of the loads upon it. Experience teaches that the amount of this deflection and curvature is very small. A beam is said to be fixed at one end when it is so arranged that the tangent to the elastic curve at that end always remains horizontal. This may be done in practice by firmly building one end into a wall. A beam fixed at one end and unsupported at the other is a cantilever.

The loads on beams are either uniform or concentrated. A uniform load embraces the weight of the beam itself and any load evenly spread over it. Uniform loads are estimated by their intensity per unit of length of the beam, and usually in pounds per linear foot. The uniform load per linear unit is designated by w, then wx will represent the load over any distance x. If l be the length of the beam the total uniform load is wl which may be represented by W. A concentrated load is a weight applied at a definite point and is designated by P.

In this chapter cantilevers and simple beams will be principally discussed, although all the fundamental principles and methods hold good for restrained and continuous beams as well. Unless otherwise stated the beams will be regarded as of uniform cross-section throughout, and in computing their weights the rules of Art. 1 will be found of service.

Prob. 28. Find the diameter of a round steel bar which weighs 12 pounds per linear foot.

Art. 14. Reactions of the Supports.

The points upon which a beam is supported react upward against the beam an amount equal to the pressure of the beam upon them. The beam, being at rest, is a body in equilibrium under the action of a system of forces which consist of the downward loads and the upward reactions. The loads are usually given in intensity and position and it is required to find the reactions. This is effected by the application of the fundamental conditions of static equilibrium, which for a system of vertical forces, are,

$$\Sigma \text{ of all vertical forces} = 0,$$

$$\Sigma \text{ of moments of all forces} = 0.$$

The first of these conditions says that the sum of all the loads on the beam is equal to the sum of the reactions. Hence if there be but one support, the condition gives at once the reaction.

For two supports the second condition must be used in connection with the first. The center of moments may be taken anywhere in the plane, but it is more convenient to take it at one of the supports. For example, consider a single concentrated load P situated at 4 feet from the left end of a simple beam whose span is 13 feet. The equation of moments, with the center at the left support, is $13\ R_2 - 4\ P = 0$, from which $R_2 = \frac{4}{13}P$. Again the equation of moments, with the center at the right support, is $13\ R_1 - 9\ P = 0$, from which $R_1 = \frac{9}{13}P$. As a check it may be observed that $R_1 + R_2 = P$.

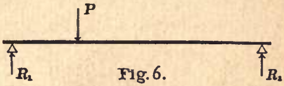


Fig. 6.

For a uniform load over a simple beam it is evident, without applying the conditions of equlibrium, that each reaction is one-half the load.

For an overhanging beam uniformly loaded, as shown in Fig. 7, let w be the load per linear unit. For a center of moments at the right support the second condition gives,

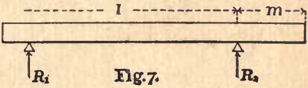


Fig. 7.

$$R_1.l - wl.\frac{l}{2} + wm.\frac{m}{2} = 0,$$

while with the center at the left support,

$$R_2.l - wl\frac{l}{2} - wm\left(l + \frac{m}{2}\right) = 0.$$

From these it is easy to deduce the reactions,

$$R_1 = \frac{wl}{2} - \frac{wm^2}{2l}. \qquad R_2 = \frac{wl}{2} + wm + \frac{wm^2}{2l}.$$

whose sum is equal to the total load $wl + wm$. Here, as in all cases of uniform load, the lever arms are taken to the centers of gravity of the portion considered.

When there are more than two supports the problem of finding the reactions from the principles of statics becomes indeterminate, since two conditions of equilibrium are only sufficient to determine two unknown quantities. By introducing, however, the elastic properties of the material, the reactions of continuous beams may be deduced as will be explained in the next chapter.

Prob. 29. Three men carry a stick of timber, one taking hold at one end and the other two at a common point. Where should this point be so that each may bear one-third of the weight?

Prob. 30. A simple beam weighing 30 pounds per linear foot is 18 feet long. A weight of 700 pounds is placed 5 feet from the left end and one of 500 pounds at 10 feet from the left end. Find the reactions due to the total load.

Art. 15. External Forces and Internal Stresses.

The external loads and reactions on a beam maintain their equilibrium by means of internal stresses which are generated in it. It is required to determine the relations between the external forces and the internal stresses.

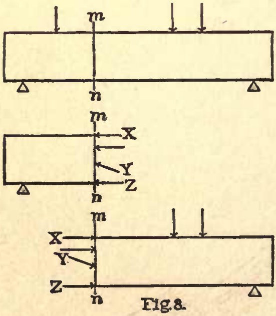


Fig. 8.

Consider a beam of any kind loaded in any manner. Imagine a plane *mn* cutting the beam at any cross-section. In that section there are acting unknown stresses of various intensities and directions. Let the beam be imagined to be separated into two parts by the cutting plane and let forces *X*, *Y*, *Z*, etc., equivalent to the internal stresses, be applied to the section as shown in Fig. 8. Then the equilibrium of each part of the beam will be undisturbed, for each part will be acted upon by a system of forces in equilibrium.

Hence the following fundamental principle is established,

> The internal stresses in any cross-section of a beam hold in equilibrium the external forces on each side of that section.

This is the most important principle in the theory of flexure. It applies to all beams, whether the cross-section be uniform or variable and whatever be the number of the spans or the nature of the loading.

Thus in the above figure the internal stresses X, Y, Z, etc., hold in equilibrium the loads and reactions on the left of the section, and also those on the right. Considering one part only a system of forces in equilibrium is seen, to which the three necessary and sufficient conditions of statics apply, namely,

$$\Sigma \text{ of all horizontal components} = 0,$$
$$\Sigma \text{ of all vertical components} = 0,$$
$$\Sigma \text{ of moments of all forces} = 0.$$

From these conditions can be deduced three laws concerning the unknown stresses in any section. Whatever be the intensity and direction of these stresses, let each be resolved into its horizontal and vertical components. The vertical components will add together and form a certain resultant force V which tends to shear off the section from the one adjacent to it. The horizontal components will be applied at different points of the cross-section, some acting in one direction and some in the other, or in other words, some of the horizontal stresses are tensile and some compressive. Hence for any section of any beam the following laws concerning the internal stresses may be stated.

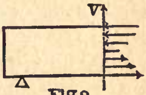


Fig. 9.

1st. The algebraic sum of the horizontal stresses is zero; or the sum of the horizontal tensile stresses is equal to the sum of the horizontal compressive stresses.

2nd. The algebraic sum of the vertical stresses forms a resultant shear which is equal to the algebraic sum of the external vertical forces on either side of the section.

3rd. The algebraic sum of the moments of the internal stresses is equal to the algebraic sum of the moments of the external forces on either side of the section.

These three theoretical laws are the foundation of the theory of the flexure of beams. Their expression may be abbreviated by introducing the following definitions.

'Resisting shear' is the name given to the algebraic sum of the vertical stresses in any section, and 'vertical shear' is the name for the algebraic sum of the external vertical forces on the left of the section. 'Resisting moment' is the name given to the algebraic sum of the moments of the internal horizontal stresses with reference to a point in the section, and 'bending moment' is the name for the algebraic sum of the moments of the external forces on the left of the section with reference to the same point. Then the three laws may be thus expressed for any section of any beam,

Sum of tensile stresses = Sum of compressive stresses.
Resisting shear = Vertical shear.
Resisting moment = Bending moment.

The second and third of these equations furnish the fundamental formulas for investigating beams. They state the relations between the internal stresses in any section and the external forces on the left-hand side of that section.

Prob. 31. A wooden beam 12 × 14 inches and 6 feet long is supported at one end by a force of 560 pounds acting at an angle of 60 degrees with the vertical, and at the other end by a vertical force Y and a horizontal force X. Find the values of X and Y.

Art. 16. The Vertical Shear.

Vertical Shear is the name given to the algebraic sum of all external forces on the left of the section considered. Let it be denoted by V, then for any section,

$V =$ Reactions on left of section minus all loads on left.

Here upward forces are regarded as positive and downward forces as negative. V is hence positive or negative according as the reaction exceeds or is less than the loads on the left of the section. To illustrate, consider a simple beam or cantilever loaded in any manner and cut at any section by a vertical plane mn. Let R be the left and R' the right reaction. Let ΣP denote the sum of all the loads on the left of the section and $\Sigma P'$ the sum of those on the right. Then, from the definition,

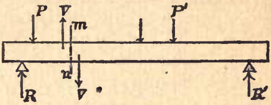


Fig. 10.

$$V = R - \Sigma P.$$

Since $R + R' = \Sigma P + \Sigma P'$ it is clear if $R - \Sigma P = + V$ that $R' - \Sigma P' = - V$, or that the resultant of all the external forces on one side of the section is equal and opposite to the resultant of those on the other side. They form, in short, a pair of shears acting very near together on either side of the section and tending to cause a sliding or detrusion along the section. The value of the vertical shear for any section of a simple beam or cantilever is readily found by the above equation. When R exceeds ΣP, the vertical shear V is positive, and the left part of the beam tends to slide upward relative to the right part. When R is less than ΣP, the vertical shear V is negative, and the left part tends to slide downward relative to the other.

The vertical shear varies greatly in value at different sections of a beam. Consider first a simple beam l feet long and weighing w pounds per linear foot. Each reaction is then $\frac{1}{2}wl$. Pass a plane at any distance x from the left support, then from the definition the vertical shear for that section is $V = \frac{1}{2}wl - wx$. Here it is seen that V has its greatest value $\frac{1}{2}wl$ when $x = 0$, that V decreases as x increases, and that V becomes 0 when $x = \frac{1}{2}l$. When x is greater than $\frac{1}{2}l$, V is negative and becomes $-\frac{1}{2}wl$ when $x = l$. The equation $V = \frac{1}{2}wl - wx$ is indeed the equation of a straight line, the origin being at the left support, and may be plotted so that the ordinate at any point of the beam will represent the vertical shear for that point, as shown in Fig. 11.

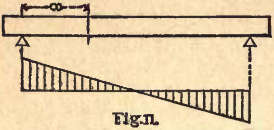


Fig. 11.

Consider again a simple beam loaded with several weights P_1, P_2, P_3, etc., and let the weight of the beam itself be neglected. Here for any point a, between the support and the first load, $V = R$; for b, $V = R - P_1$; for c, $V = R - P_1 - P_2$, and so on. For the case of three loads the graphical representation of vertical shears is shown in Fig. 12.

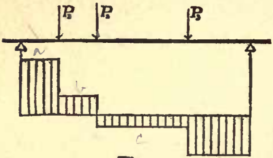


Fig. 12.

For a cantilever beam there is no reaction at the left end, and for any section, $V = -\Sigma P$. In any case ΣP must include both uniform and concentrated loads if such are upon the beam. For a cantilever with two concentrated loads and a uniformly distributed load the vertical shear for a is $V = -P_1 - wx$, and for b, $V = -P_1 - P_2 - wx$, where x is the distance from the left-hand end, and the graphical representation is shown in Fig. 13. Here V is a maximum at the wall and then immediately becomes 0.

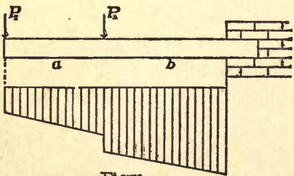


Fig. 13.

Prob. 32. A simple beam 12 feet long and weighing 20 pounds per linear foot has a load of 600 pounds at 2 feet from the left end. Find the vertical shears at the ends and under the load. Draw a diagram to show the distribution of vertical shears.

ART. 17. THE BENDING MOMENT.

Consider again any beam cut at any section by a vertical plane. The third fundamental condition of static equilibrium (Art. 15) states that the algebraic sum of the moments of the external forces on one side of the section is equal to the sum of the resisting moments of the internal stresses in the section. This is true for any center of moments.

Bending moment is the name given to the algebraic sum of the moments of the external forces on the left of the section with reference to a point in that section. Let it be denoted by M. Then, for a cantilever or simple beam,

$M =$ moment of reaction minus sum of moments of loads.

Here the moment of upward forces is taken as positive and that of downward forces as negative. M may hence be positive or negative according as the first or second term is the greater.

For a simple beam of length l, uniformly loaded, each reaction is $\frac{1}{2}wl$. For any section distant x from the left support the bending moment is $M = \frac{1}{2}wl.x - wx.\frac{1}{2}x$, or $M = \frac{1}{2}w(lx - x^2)$. Here $M = 0$ when $x = 0$ and also when $x = l$, and M is a maximum when $x = \frac{1}{2}l$. The equation, in short, is that of a parabola whose maximum ordinate is $\frac{wl^2}{8}$ and whose graphical representation is as given in Fig. 14, each ordinate showing the value of M for the corresponding value of the abscissa x.

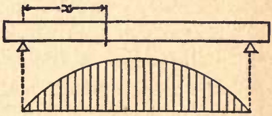


Fig. 14.

Consider next a simple beam loaded with only three weights P_1, P_2 and P_3. Here for any point between the left support and the first load $M = Rx$, and for any point between the first and second loads $M = Rx - P_1(x - a)$. Each of these expressions is the equation of a straight line, x being the abscissa and M the ordinate, and it is easy to see that the graphical representation of bending moments is as shown in Fig. 15.

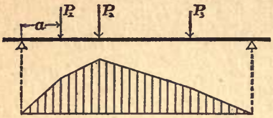


Fig. 15.

For a cantilever there is no reaction at the left end and all the bending moments are negative. For instance, for a cantilever uniformly loaded and having a load at the end the bending moment is $M = -Px - \frac{1}{2}wx^2$. Here the variation of moments may be represented by a parabola, M being 0 at the free end and a maximum at the wall.

For any given case the bending moment at any section may be readily found by using the definition given above. The bending moment in all cases is a measure of the tendency of the external forces on the left of the section to turn the beam around a point in that section. This turning is prevented by the internal resisting moments of the stresses in the section, whose sum exactly equals the bending moment.

The bending moment is a compound quantity resulting from the multiplication of a force by a distance. Usually the forces are expressed in pounds and the distances in feet or inches; then the bending moments are pound-feet or pound-inches. Thus if a load of 500 pounds be at the middle of a simple beam of 8 feet span, the bending moment under the load is,

$$M = 250 \times 4 = 1\,000 \text{ pound-feet} = 12\,000 \text{ pound-inches.}$$

Again let a simple beam of 8 feet span be uniformly loaded with 500 pounds. Then the bending moment at the middle is,

$$M = 250 \times 4 - 250 \times 2 = 500 \text{ pound-feet} = 6\,000 \text{ pound-inches.}$$

Prob. 33. A simple beam of 12 feet span weighs 20 pounds per linear foot and has a load of 250 pounds at 3 feet from the left end. Find the bending moments for the quarter points and for the middle of the beam.

Prob. 34. A beam 6 feet long and weighing 20 pounds per foot is placed upon a single support at its center. Compute the bending moments for sections distant 1, 2, 3 and 4 feet from the left end, and draw a curve to show the distribution of moments throughout the beam.

ART. 18. THEORETICAL AND EXPERIMENTAL LAWS.

From the three necessary conditions of static equilibrium, as stated in Art. 15, three important theoretical laws regarding internal stresses were deduced. These alone, however, are not sufficient for the full investigation of the subject, but recourse must be had to experience and experiment. Experience teaches that when a beam deflects one side becomes concave and the other convex, and it is reasonable to suppose that the horizontal tensile stresses are on the convex side and the compressive stresses on the concave. By experiments on beams this is confirmed and the following laws deduced.

(*F*)—The horizontal fibers on the convex side are elongated and those on the concave are shortened, while near the center is a neutral surface which is unchanged in length.

(*G*)—The amount of elongation or compression of any fiber is directly proportional to its distance from the neutral axis. Hence by law (*B*) the horizontal stresses are also directly proportional to their distances from the neutral axis, provided the elastic limit of the material be not exceeded.

The neutral surface passes through the centers of gravity of the cross-sections. To prove this let a be the area of any elementary fiber and z its distance from the neutral surface. Let S be the unit-stress on the fiber most remote from the neutral surface at the distance c. Then by law (*G*),

$$\frac{S}{c} = \text{unit-stress at the distance unity,}$$

$$\frac{S}{c}z = \text{unit-stress at the distance } z,$$

therefore $\frac{S}{c}az$ = the total stress on any fiber of area a,

and $\Sigma\frac{Saz}{c}$ = algebraic sum of all horizontal stresses.

But by the first law of Art. 15 this algebraic sum is zero, and since S and c are constants $\Sigma az = 0$. This however is the condition which makes the line of reference pass through the center of gravity as shown in elementary mechanics. Therefore the neutral surface of beams passes through the centers of gravity of the cross-section.

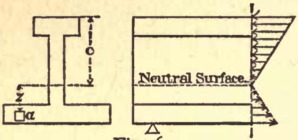


Fig. 16.

The 'neutral axis' of a cross-section is the line in which the neutral surface intersects the plane of the cross-section. On the left of Fig. 16 is shown the neutral axis of a cross-section and on the right a trace of the neutral surface.

Prob. 35. A beam 3 inches wide and 6 inches deep is strained so that the unit-stress at the remotest fiber of a certain cross-section is 600 pounds per square inch. Find the sum of all the tensile stresses on the cross-section.

Art. 19. The Resisting Shear and the Resisting Moment.

The resisting shear is the algebraic sum of all the vertical components of the internal stresses at any section of the beam. If A be the area of that section and S_s the shearing unit-stress, regarded as uniform over the area, then from formula (1),

$$\text{Resisting shear} = AS_s.$$

The resisting moment is the algebraic sum of the moments of the internal horizontal stresses at any section with reference to a point in that section. To find an expression for its value let S be the horizontal unit-stress, tensile or compressive as the case may be, upon the fiber most remote from the neutral axis and let c be the shortest distance from that fiber to said axis. Also let z be the distance from the neutral axis to any fiber having the elementary area a. Then by law (G) and Fig. 16,

$$\frac{S}{c} = \text{unit-stress at a distance unity,}$$

$$\frac{S}{c}z = \text{unit-stress at distance } z,$$

$$\therefore \quad \frac{aSz}{c} = \text{total stress on area } a,$$

$$\text{and} \quad \frac{aSz^2}{c} = \text{moment of this stress about neutral axis.}$$

$$\therefore \quad \Sigma\frac{aSz^2}{c} = \text{resisting moment of horizontal stresses.}$$

Since S and c are constants this expression may be written $\frac{S}{c}\,\Sigma az^2$. But Σaz^2, being the sum of the products formed by multiplying each elementary area by the square of its distance from the neutral axis, is the moment of inertia of the cross-section with reference to that axis and may be denoted by I. Therefore,

$$\text{Resisting moment} = \frac{SI}{c}.$$

Prob. 36. A wooden beam 6 × 12 inches and five feet long is supported at one end and kept level by two horizontal forces X and Y acting at the other end in the median line of the cross-section, the former at 2 inches from the top and the latter at 2 inches from the base. Find the values of X and Y.

ART. 20. THE TWO FUNDAMENTAL FORMULAS.

Consider again any beam loaded in any manner and cut at any section by a vertical plane. The internal stresses in that section hold in equilibrium the external forces on the left of the section, and as shown in Art. 15,

$$\text{Resisting shear} = \text{Vertical shear},$$
$$\text{Resisting moment} = \text{Bending moment}.$$

In the last article values of the resisting shear and the resisting moments were deduced. These two equations then become,

$$(3) \qquad S_sA = V,$$

$$(4) \qquad \frac{SI}{c} = M,$$

which are the fundamental formulas for investigating the strength of beams. From (3) the average shearing unit-stress S_s at any cross-section may be found. From (4) the tensile or compressive unit-stress S upon the remotest fiber at any cross-section may be computed. It is now seen that transverse stresses are investigated by resolving them into the simple stresses of tension, compression and shear. Whether S be tension or compression depends, in any particular case, upon whether c is measured to the concave or convex side of the beam.

V is readily found, as explained in Art. 16, for any given section of any beam. Usually only its maximum values are needed in investigations of strength and these will be found at the supports.

M is readily found, as explained in Art. 17, for any given section. Usually only its maximum values need be determined and these are near the middle for a simple beam and at the support for a cantilever. The values of c and I for any given cross-section must be computed by the well-known methods explained in elementary mechanics. Then the unit-stress S will be found from (4).

Experience and experiment teach us that simple beams of uniform section break near the middle by the tearing or crushing of the fibers and very rarely at the supports by shearing. Hence it is formula (4) that is mainly needed in the practical investigation of beams. The following example and problem relate to formula (3) only, while formula (4) will receive detailed discussion in the subsequent articles.

As an example, consider a wrought iron **I** beam 15 feet long and weighing 200 pounds per yard, over which roll two locomotive wheels 6 feet apart and each bearing 12 000 pounds. The maximum vertical shear at the left support will evidently occur when one wheel is at the support. The reaction will then be $500 + 12\,000 + \frac{9}{15} \times 12\,000 = 19\,700$ pounds. The greatest value of V in the beam is then 19 700 pounds. As the area of the cross-section is 20 square inches the average shearing unit-stress is 985 pounds so that the factor of safety is about 50.

Prob. 37. A wooden beam 6×9 inches and 12 feet in span carries a uniform load of 20 pounds per foot besides its own weight and also two wagon wheels one weighing 4 000 pounds and the other 3 000 pounds. Find the factor of safety against shearing.

ART. 21. CENTER OF GRAVITY OF CROSS-SECTIONS.

The fundamental formula (4) contains c, the shortest distance from the remotest part of the cross-section to a horizontal axis passing through the center of gravity of that cross-section. The methods of finding c are explained in books on theoretical mechanics and will not here be repeated. Its values for some of the simplest cases are however recorded for reference.

For a rectangle whose height is d, $c = \frac{1}{2}d$.

For a circle whose diameter is d, $c = \frac{1}{2}d$.

For a triangle whose altitude is d, $c = \frac{2}{3}d$.

For a square with side d having one diagonal vertical, $c = d\sqrt{\frac{1}{2}}$

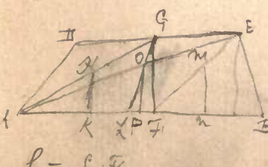
For a **I** whose depth is d, $c = \frac{1}{2}d$.

For a **⊥** whose depth is d, thickness of flange t, width of flange b, and thickness of web t'

$$c = \frac{\frac{1}{2}t'd^2 + t(b-t')(d-\frac{1}{2}t)}{t'd + t(b-t')}.$$

For a trapezoid whose depth is d, upper base b, and lower base b', $c = \frac{b + 2b'}{b + b'} \cdot \frac{d}{3}$.

The student should be prepared to readily apply the principle of moments to the deduction of the numerical value of c for any



given cross-section. In nearly all cases the given area may be divided into rectangles, triangles, and circular areas, whose centers of gravity are known, so that the statement of the equation for finding c is very simple.

Prob. 38. Find the value of c for a rail headed beam whose section is made up of a rectangular flange $\frac{3}{4} \times 4$ inches, a rectangular web $\frac{1}{2} \times 5$ inches, and an elliptical head $\frac{3}{4}$ inches deep and $1\frac{1}{2}$ inches wide.

Art. 22. Moment of Inertia of Cross-Sections.

The fundamental formula (4) contains I, the moment of inertia of the cross-section of the beam with reference to a horizontal axis passing through the center of gravity of that cross-section. Methods of determining I are explained in works on elementary mechanics and will not here be repeated, but the values of some of the most important cases are recorded for reference.

For a rectangle of base b and depth d, $\quad I = \frac{bd^3}{12}.$

For a circle of diameter d, $\quad I = \frac{\pi d^4}{64}.$

For an ellipse with axes a and b, the latter vertical, $\quad I = \frac{\pi ab^3}{64}.$

For a triangle of base b and depth d, $\quad I = \frac{bd^3}{36}.$

For a square with side d having one diagonal vertical, $\quad I = \frac{d^4}{12}.$

For a **I** with base b, depth d, thickness of flanges t and thickness of web t',

$$I = \frac{bd^3 - (b - t')(d - 2t)^3}{12}.$$

For an ⊥ with base b, depth d, thickness of flange t, thickness of web t' and area A,

$$I = \frac{bd^3 - (b - t')(d - t)^3}{3} - Ac^2.$$

The value of I for any given section may always be computed by dividing the figure into parts whose moments of inertia are known and transferring these to the neutral axis by means of the familiar rule $I_1 = I_0 + Ah^2$, where I_0 is the primitive value, I_1 the value for any parallel axis, A the area of the figure and h the distance between the two axes.

Prob. 39. Find the moment of inertia of a triangle with reference to its base, and also with reference to a parallel axis passing through its vertex.

Prob. 40. Compute the least moment of inertia of a trapezoid whose altitude is 3 inches, upper base 2 inches and lower base 5 inches.

ART. 23. THE MAXIMUM BENDING MOMENT.

The fundamental equation (4), namely $\frac{SI}{c} = M$, is true for any section of any beam, I being the moment of inertia of that section about its neutral axis, c the vertical distance from that axis to the remotest fiber, S the tensile or compressive unit-stress on that fiber, and M the bending moment of all the external forces on one side of the section. For a beam of constant cross-section S varies directly as M, and the greatest S will be found where M is a maximum. The place where M has its maximum value may hence be called the 'dangerous section,' it being the section where the horizontal fibers are most highly strained.

For a simple beam uniformly loaded with w pounds per linear unit the dangerous section is evidently at the middle, and as shown in Art. 17, the maximum M is $\frac{wl^2}{8}$.

For a simple beam loaded with a single weight P at the distance p from the left support, the left reaction is $R = P\frac{l-p}{l}$, and the maximum moment is $\frac{P(l-p)p}{l}$. If P be movable the distance

p will be variable and when the load is at the middle the maximum M is $\frac{Pl}{4}$.

For a cantilever the dangerous section is evidently at the wall and for a uniform load the maximum M is $-\frac{wl^2}{2}$.

For a beam loaded with given weights, either uniform or concentrated, it may be shown that the dangerous section is at the point where the vertical shear passes through zero. To prove this let P be any concentrated load and p its distance from the left support, and w the uniform load per linear unit. Then, for any section distant x from the left support,

$$M = Rx - wx.\frac{x}{2} - \Sigma P(x - p).$$

To find the value of x which renders this a maximum the first derivative must be put equal to zero; thus,

$$\frac{dM}{dx} = R - wx - \Sigma P = 0.$$

But $R - wx - \Sigma P$ is the vertical shear V for the section x (see Art. 16). Therefore the maximum moment occurs at the section where the vertical shear passes through 0. To find the dangerous section an expression may be written for V in terms of x and the value of x determined when V changes sign. Thus for a simple beam uniformly loaded $V = \frac{1}{2}wl - wx = 0$, and $x = \frac{1}{2}l$. For concentrated loads it will generally be necessary to find by trial the point where the shear becomes zero.

Prob. 41. A simple beam 12 feet long weighs 20 pounds per foot and carries a load of 100 pounds at 4 feet from the left end and a load of 50 pounds at 7 feet from the left end. Find the dangerous section.

Prob. 42. A beam 25 feet long, uniformly loaded with w pounds per linear foot, is supported at the left end and at a point 5 feet from the right end. Find the two dangerous sections and the two maximum moments.

ART. 24. THE INVESTIGATION OF BEAMS.

The investigation of a beam consists in deducing the greatest horizontal unit-stress S in the beam from the fundamental formula (4). This may be written,

$$S = \frac{Mc}{I}.$$

First, from the given dimensions find, by Art. 21, the value of c and by Art. 22 the value of I. Then by Art. 23 determine the value of maximum M. From (4) the value of S is now known. Usually c and I are taken in inches, and M in pound-inches; then the value of S will be in pounds per square inch.

The value of S will be tension or compression according as the remotest fiber lies on the concave or convex side of the beam. If S' be the unit-stress on the opposite side of the beam and c' the distance from the neutral axis, then from law (G),

$$\frac{S}{c} = \frac{S'}{c'} \qquad \text{and} \qquad S' = S\frac{c'}{c}.$$

If S be tension, S' will be compression, and *vice versa*. Sometimes it is necessary to compute S' as well as S in order to thoroughly investigate the stability of the beam. By comparing the values of S and S' with the proper working unit-stresses for the given materials (see Art. 8) the degree of security of the beam may be inferred.

As an example consider a wrought iron **I** beam whose depth is 12 inches, width of flange 4.5 inches, thickness of flange 1 inch and thickness of web 0.78 inches. It is supported at its ends forming a span of 12 feet, and carries two loads each of 10 000 pounds one at the middle and the other at one foot from the right end.

By Art. 14, $R = 6193$ pounds.
By Art. 21, $c = 6$ inches.
By Art. 22, $I = 338$ inches^4.
By Art. 23, $x = 6$ feet for dangerous section.
By Art. 23, max. $M = 36\,078 \times 12$ pound-inches.

Then from formula (4) the unit-stress at the dangerous section is,

$$S = \frac{36\ 078 \times 12 \times 6}{338} = 7\ 700 \text{ pounds per square inch.}$$

This is the compressive unit-stress on the upper fiber and also the tensile unit-stress on the lower fiber, and being only about one-third of the elastic limit for wrought iron and about one-seventh of the ultimate strength it appears that the beam is entirely safe. It will usually be best in solving problems to insert all the numerical values at first in the formula and thus obtain the benefit of cancellation.

A short beam heavily loaded also needs to be investigated for the shearing stress at the supports in the manner mentioned in Art. 20, but in ordinary cases there is little danger of failure from this cause.

Prob. 43. Find the factor of safety of a simple wooden beam 2 × 4 inches when loaded at the middle with 1 000 pounds.

Prob. 44. A piece of scantling 2 inches square and 10 feet long is hung horizontally by a rope at each end and three painters stand upon it. Is it safe?

Prob. 45. A wrought iron bar one inch in diameter and two feet long is supported at its middle and a load of 500 pounds hung upon each end of it. Find its factor of safety.

Art. 25. Safe Loads for Beams.

The proper load for a beam should not make the value of S at the dangerous section greater than the allowable unit-stress. This allowable unit-stress or working strength is to be assumed according to the circumstances of the case by first selecting a suitable factor of safety from Art. 8 and dividing the ultimate strength of the material by it, the least ultimate strength whether tensile or compressive being taken. For any given beam the quantities I and c are known. Then, by the general formula (4), the bending moment M may be expressed in terms of the unknown loads on the beam, and thus those loads be found. The sign of the bending moment should not be used in (4), since that sign merely denotes whether the upper fiber of the beam is in tension or compression, or indicates the direction in which the external forces tend to bend it.

As an example, consider a cantilever projecting from a wall whose length is 6 feet, breadth 2 inches, depth 3 inches and which is loaded uniformly with w pounds per linear foot. It is required to find the value of w so that S may be 800 pounds per square inch. Here we have $c = 1\frac{1}{2}$ inches, $I = \frac{54}{12}$, and $M = 36 \times 6w$. Then from (4),

$$216w = \frac{800 \times 54}{1\frac{1}{2} \times 12}, \qquad \text{whence} \qquad w = 11 \text{ pounds.}$$

Since a wooden beam 2×3 inches weighs about 2 pounds per linear foot, the safe load in this case will be about 9 pounds per foot.

Prob. 46. A wooden beam 8×9 inches and of 14 feet span carries a load, including its own weight, of w pounds per linear foot. Find the value of w for a factor of safety of 10.

Prob. 47. A cast iron beam one inch square and of 4 feet span carries a load P at the middle. Find P so that the greatest horizontal unit-stress at the dangerous section shall be 3 000 pounds per square inch.

ART. 26. DESIGNING OF BEAMS.

When a beam is to be designed the loads to which it is to be subjected are known, as also is its length. Thus the maximum bending moment may be found. The allowable working strength S is assumed in accordance with engineering practice. Then formula (4) may be written,

$$\frac{I}{c} = \frac{M}{S},$$

and the numerical value of the second member be found. The dimensions to be chosen for the beam must then have a value of $\frac{I}{c}$ equal to this numerical value, and these in general are determined tentatively, certain proportions being first assumed. The selection of the proper proportions and shapes of beams for different cases requires much judgment and experience. But whatever forms be selected they must in each case be such as to satisfy the above equation.

For instance, a wrought iron beam of 4 feet span is required to carry 500 pounds at the middle. Here, by Art. 23, the value of maximum M is 6 000 pound-inches. From Art. 8 the value of S for a variable load is about 10 000 pounds per square inch. Then,

$$\frac{I}{c} = \frac{6\ 000}{10\ 000} = 0.6 \text{ inches}^3.$$

An infinite number of cross-sections may be selected with this value of $\frac{I}{c}$. If the beam is to be round and of diameter d, it is known that $c = \frac{1}{2}d$ and $I = \frac{\pi d^4}{64}$. Hence,

$$\frac{\pi d^3}{32} = 0.6, \qquad \text{whence} \qquad d = 1.83 \text{ inches.}$$

If the cross-section is to be rectangular, the dimensions 1×2 inches would give the value of $\frac{I}{c}$ as $\frac{2}{3}$ which would be a little too large, while $1 \times 1\frac{3}{4}$ would give $\frac{I}{c}$ as about 0.53 which would be too small.

Sometimes cross-sections not symmetrical to the neutral axis are designed, particularly for cast iron where the compressive unit-stress may be taken greater than the tensile. In no case of design however should the dimensions be so selected as to render the unit-stress on either side greater than the elastic limit of the material.

Prob. 48. A rectangular beam of 14 feet span carries a load of 1 000 pounds at its center. If its width is 4 inches find its depth for a factor of safety of 10.

Prob. 49. Design a hollow circular wrought iron beam for a span of 12 feet to carry a load of 320 pounds per linear foot.

ART. 27. THE MODULUS OF RUPTURE.

The fundamental formula (4) is only true for stresses within the elastic limit, since beyond that limit the law (G) does not hold, and the horizontal unit-stresses are no longer proportional to their distances from the neutral axis, but increase in a more rapid ratio as shown in the sketch. It is however very customary to apply (4) to the rupture of beams.

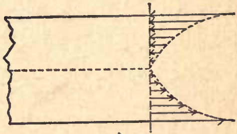


Fig. 17.

The 'modulus of rupture' is the value of S deduced from formula (4) when the beam is loaded up to the breaking point. It is always found by experiment that the modulus of rupture does not agree with either the ultimate tensile or compressive strength of the material but is intermediate between them. If formula (4) were valid beyond the elastic limit, the value of S for rupture would agree with the least ultimate strength, with tension in the case of cast iron and with compression in the case of timber. The modulus of rupture is denoted by S_r.

The average values of the modulus rupture are given in the following table, which also contains the average ultimate tensile and compressive strengths, previously stated in Arts. 5 and 6, all in pounds per square inch.

Material.	Tensile Strength, S_t.	Modulus of Rupture, S_r.	Compressive Strength, S_c.
Timber	10 000	9 000	8 000
Brick			2 500
Stone		2 000	6 000
Cast Iron	20 000	35 000	90 000
Wrought Iron	55 000	55 000	55 000
Steel	100 000	120 000	150 000

By the use of the experimental values of the modulus of rupture it is easy with the help of formula (4) to determine what load will cause the rupture of a given beam, or what must be its length or size in order that it may rupture under assigned loads.

Prob. 50. A wooden beam 4×6 inches and 9 feet span has a load P at the middle. Find the value of P to break it.

Prob. 51. A wrought iron cantilever 2 inches square projects from a wall. Find its length in order to rupture under its own weight.

Prob. 52. What must be the size of a square wooden beam of 8 feet span in order to break under its own weight?

ART. 28. COMPARATIVE STRENGTHS.

The strength of a beam is measured by the load that it can carry. Let it be required to determine the relative strength of the four following cases.

1st, A cantilever loaded at the end with W,
2nd, A cantilever uniformly loaded with W,
3rd, A simple beam loaded at the middle with W,
4th, A simple beam loaded uniformly with W.

Let l be the length in each case. Then, from Art. 23 and formula (4),

For 1st, $M = Wl$ and hence $W = \frac{SI}{lc}$.

For 2nd, $M = \frac{Wl}{2}$ and hence $W = 2\frac{SI}{lc}$.

For 3rd, $M = \frac{Wl}{4}$ and hence $W = 4\frac{SI}{lc}$.

For 4th, $M = \frac{Wl}{8}$ and hence $W = 8\frac{SI}{lc}$.

Therefore the comparative strengths of the four cases are as the numbers 1, 2, 4, 8. That is, if four such beams be of equal size and length and of the same material, the 2nd is twice as strong as the 1st, the 3rd four times as strong, and the 4th eight times as strong.

These equations also show that the strength of a beam varies directly as S, inversely as the length l, and directly as $\frac{I}{c}$. They also prove that a uniform load produces only one-half as much stress as an equivalent concentrated load. These general laws of the strength of beams are very important.

Prob. 53. Compare the strength of a rectangular beam 2 inches wide and 4 inches deep with that of a circular beam 3 inches in diameter.

Prob. 54. Compare the strength of a wooden beam 4 × 6 inches and 10 feet span with that of a wrought iron beam 1 × 2 inches and 7 feet span.

ART. 29. RECTANGULAR BEAMS.

For a beam of rectangular cross-section let b be the breadth and d the depth. Then the formula (4) becomes,

$$\frac{Sbd^2}{6} = M.$$

For cantilevers and common beams let W be the load, either uniform or concentrated as in Art. 28, then,

$$W = n\frac{Sbd^2}{6l},$$

where n is either 1, 2, 4 or 8, as the case may be.

This equation shows the important laws that the strength of a rectangular beam varies directly as its breadth, directly as the square of its depth and inversely as its length. The reason why rectangular beams are put with the greatest dimension vertical is now apparent.

To find the strongest rectangular beam that can be cut from a circular log of given diameter D, it is necessary to make bd^2 a maximum. Or the value of b is to be found which makes $b(D^2 - b^2)$

a maximum. By placing the first derivative equal to zero this value of b is readily found. Thus,

$$b = D\sqrt{1/3} \qquad \text{and} \qquad d = D\sqrt{2/3}.$$

Hence very nearly, $b : d :: 5 : 7$. From this it is evident that the way to lay off the strongest beam on the end of a circular log is to divide the diameter into three equal parts, from the points of division draw perpendiculars to the circumference, and then join the points of intersection with the ends of the diameter, as shown in the figure.

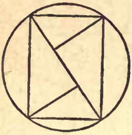


Fig. 18.

Prob. 55. Compare the strength of a cylindrical beam with that of the strongest rectangular beam that can be cut from it.

Art. 30. Wrought Iron I Beams.

Wrought iron I beams are rolled at present in about thirteen sizes or different depths. Of each size there is a light and a heavy weight, and by giving special orders weights intermediate in value may be obtained. They are extensively used in engineering and architecture. The following table gives the sizes, weights and moments of inertia of those manufactured by Carnegie Bros. & Co., Pittsburgh, Pa. The sizes of different manufacturers agree as to depth, but vary slightly with regard to proportions of cross-section, weights per foot, and moments of inertia. Fig. 19 shows the proportions of the light and heavy 6 inch beams.

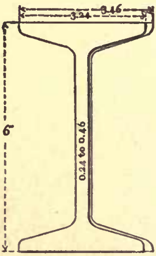


Fig. 19.

The moments of inertia in the fourth column of the table are taken about an axis perpendicular to the web at the center, this being the neutral axis of the cross-section when used as a beam. The values of I' are for use in Chapter V. In investigating the strength of a given I beam the value of $\frac{I}{c}$ is taken from the table and S is computed from formula (4). In designing an **I** beam for a given span and loads the value of $\frac{I}{c}$ is found by (4) from the data and then from the table that **I** is selected which has the nearest or next highest corresponding value.

Size. Depth. Inches.	Width of Flange. Inches.	Weight per foot. Pounds.	I. Inches^4.	$\frac{I}{c}$ Inches^3.	I'. Inches^4.
Heavy 15	5.81	80	750	100.	29.9
Light 15	5.55	67	677	90.3	25.4
H 15	5.33	65	614	81.9	20.0
L 15	5.03	50	530	70.6	16.3
H 12	5.09	60	340	56.7	15.5
L 12	4.64	42	275	45.9	11.0
H 10½	4.92	45	201	38.3	10.7
L 10½	4.54	31½	165	31.4	8.01
H 10	4.77	45	187	37.5	11.3
L 10	4.32	30	150	30.0	7.94
H 9	4.33	33	117	26.0	7.14
L 9	4.01	23½	97.5	21.7	5.48
H 8	4.29	35	90.4	22.6	6.96
L 8	3.81	22	69.6	17.5	4.57
H 7	3.91	25	54.3	15.5	4.87
L 7	3.61	18	45.8	13.1	3.72
H 6	3.46	18	28.4	9.48	2.51
L 6	3.24	13½	24.5	8.16	2.00
H 5	2.91	13	14.2	5.69	1.34
L 5	2.73	10	12.3	4.94	1.08
H 4	2.63	10	6.99	3.50	0.87
L 4	2.48	8	6.19	3.10	0.71
H 3	2.52	9	3.54	2.36	0.84
L 3	2.32	7	3.09	2.06	0.55

For example, let it be required to determine which **I** should be selected for a floor loaded with 150 pounds per square foot, the beams to be of 20 feet span and spaced 12 feet apart between centers, and the maximum unit-stress S to be 12 000 pounds per square inch. Here the total uniform load on the beam is

$12 \times 20 \times 150 = 36\,000$ pounds $= W$. From formula (4),

$$\frac{I}{c} = \frac{M}{S} = \frac{36\,000 \times 20 \times 12}{8 \times 12\,000} = 90.$$

and hence from the table, the light 15 inch **I** should be selected.

Steel **I** beams and the other shapes are now beginning to be used, and will undoubtedly be very common in a few years.

Prob. 56. A heavy 15 inch **I** beam of 12 feet span sustains a uniformly distributed load of 41 net tons. Find its factor of safety. Also the factor of safety for a 24 feet span under the same load.

Prob. 57. A floor, which is to sustain a uniform load of 175 pounds per square foot, is to be supported by heavy 10 inch **I** beams of 15 feet span. Find their proper distance apart from center to center so that the maximum fiber stress may be 12 000 pounds per square inch.

Prob. 58. What **I** should be selected for this floor if the beams are to be spaced 3 feet 7 inches from center to center.

Art. 31. Wrought Iron Deck Beams.

Deck beams are used in the construction of buildings, and are of a section such as shown in figure 20. The heads are formed with arcs of circles but may be taken as elliptical in computing the values of c and I. The following table gives dimensions of deck beams manufactured by Carnegie Bros. & Co.

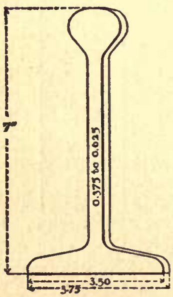


Fig 20.

By means of formula (4) a given deck beam may be investigated or safe loads be determined for it, or one may be selected for a given load and span. Sometimes **T** irons are used instead of deck beams; the values of c and I for these may be computed with sufficient accuracy by regarding the web and flanges as rectangular as in Arts. 21 and 22.

Size. Depth. Inches.		Width of Flange. Inches.	Thickness of Web. Inches.	Weight per ft. P'nds.	c. Inches.	I. Inch's⁴.	$\frac{I}{c}$. Inch's³.
Heavy	9	3.97	0.625	30	4.59	91.9	20.0
Light	9	3.75	0.406	23½	4.60	78.6	17.1
H	8	4.00	0.750	28	4.49	63.3	14.1
L	8	3.75	0.500	21½	4.58	52.1	11.6
H	7	3.75	0.625	23	3.98	43.0	10.8
L	7	3.50	0.375	17	4.00	34.4	8.6

Prob. 59. A heavy 7″ deck beam is loaded uniformly with 50 000 pounds. Find its factor of safety for a span of 22 feet.

ART. 32. CAST IRON BEAMS.

Wrought iron beams are usually made with equal flanges since the resistance of wrought iron is about the same for both tension and compression. For cast iron, however, the flange under tension should be larger than that under compression, since the tensile resistance of the material is much less than its compressive resistance. Let S' be the unit-stress on the remotest fiber on the tensile side and S that on the compressive side, at the distances c' and c respectively from the neutral axis. Then, from law (G),

$$\frac{c}{c'} = \frac{S}{S'}.$$

Now if the working values of S and S' can be selected the ratio of c to c' is known and a cross-section can be designed, but it is difficult to assign these proper values on account of our lack of knowledge regarding the elastic limits of cast iron.

According to Hodgkinson's investigations the following are dimensions for a cast iron beam of equal ultimate strength.

Thickness of web	=	t,
Depth of beam	=	$13.5t$,

Width of tensile flange	=	$12t$,
Thickness of tensile flange	=	$2t$,
Width of compressive flange	=	$5t$,
Thickness of compressive flange	=	$1\frac{1}{3}t$,
Value of c	=	$9t$,
Value of I	=	$923t^4$.

Here the unit-stress in the tensile flange is one-half that in the the compressive flange. Although these proportions may be such as to allow the simultaneous rupture of the flanges, yet it does not necessarily follow that they are the best proportions for ordinary working stresses, since the factors of safety in the flanges as computed by the use of formula (4) would be quite different. The proper relative proportions of the flanges of cast iron beams for safe working stresses have never been definitely established, and on account of the extensive use of wrought iron the question is not now so important as formerly.

As an illustration of the application of formula (4) let it be required to determine the total uniform load W for a cast iron ⊥ beam of 14 feet span, so that the factor of safety may be 6, the depth of the beam being 18 inches, the width of the flange 12 inches, the thickness of the stem 1 inch, and the thickness of the flange $1\frac{1}{4}$ inches. First, from Art. 21 the value of c is found to be 12.63 inches, and that of c' to be 5.37 inches. From Art. 22 the value of I is computed to be 1 031 inches^4. From Art. 23 the bending moment is,

$$M = \frac{wl}{8} = 21\,W \text{ pound-inches.}$$

Now with a factor of safety of 6 the working strength S on the the remotest fiber of the stem at the dangerous section is to be $\frac{90\,000}{6}$ pounds per square inch. Hence from formula (4),

$$21\,W = \frac{90\,000 \times 1\,031}{6 \times 12.63}, \quad \text{whence} \quad W = 58\,300 \text{ pounds.}$$

Again with a factor of safety of 6 the working strength S' on the remotest fiber of the flange at the dangerous section is to be $\frac{20\,000}{6}$ pounds per square inch. Hence from the formula,

$$21\,W = \frac{20\,000 \times 1\,031}{6 \times 5.37}, \quad \text{whence} \quad W = 30\,400 \text{ pounds.}$$

The total uniform load on the beam should hence not exceed 30 400 pounds. Under this load the factor of safety on the tensile side is 6, while on the compressive side it is nearly 12.

Prob. 60. A cast iron beam in the form of a channel, or hollow half rectangle, is often used in buildings. Suppose the thickness to be uniformly one inch, the base 8 inches, the height 6 inches and the span 12 feet. Find the values of S and S' at the dangerous section under a uniform load of 15 net tons.

ART. 33. GENERAL EQUATION OF THE ELASTIC CURVE.

When a beam bends under the action of exterior forces the curve assumed by its neutral surface is called the elastic curve. It is required to deduce a general expression for its equation.

Let *pu* in the figure be any normal section in any beam. Let *mn* be any short length *dl*, and draw *vmq* parallel to the normal section through *n*. Previous to the bending the sections *pu* and *st* were parallel; now they intersect at *o* the center of curvature. Previous to the bending *ps* was equal to *dl*, now it has elongated the amount *pq* or λ. The distance *mp* is the quantity c. The elongation λ is produced by the unit-stress S, and from (2) its value is,

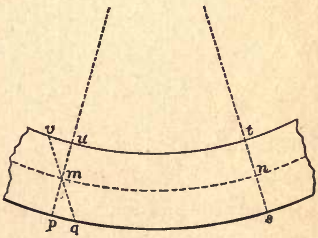


Fig. 21.

$$\lambda = \frac{S\,dl}{E},$$

where E is the coefficient of elasticity of the material of the beam. From the similar figures *omn* and *mpq*,

$$\frac{om}{mn} = \frac{mp}{pq}, \qquad \text{or} \qquad \frac{R}{dl} = \frac{c}{\lambda},$$

where R is the radius of curvature *om*. Inserting in this the above value of λ, it becomes,

$$\frac{S}{c} = \frac{E}{R}.$$

But from the fundamental formula (4),

$$\frac{S}{c} = \frac{M}{I},$$

and hence, by comparison,

$$M = \frac{EI}{R}.$$

This is the formula which gives the relation between the bending moment of the exterior forces and the radius of curvature at any section.

Now, in works on the differential calculus, the following value is deduced for the radius of curvature of any plane curve whose abscissa is x, ordinate y and length l, namely,

$$R = \frac{dl^3}{dx.d^2y}.$$

Hence the most general equation of the elastic curve is,

$$\frac{dl^3}{dx.d^2y} = \frac{EI}{M},$$

which applies to the flexure of all bodies governed by the laws of Arts. 3 and 18.

In discussing a beam the axis of x is taken as horizontal and that of y as vertical. Experience teaches us that the length of a small part of a bent beam does not materially differ from that of its horizontal projection. Hence dl may be placed equal to dx for all beams, and the above equation reduces to the form,

$$\frac{d^2y}{dx^2} = \frac{M}{EI}. \qquad (5)$$

This is the general equation of the elastic curve, applicable to all beams whatever be their shapes, loads or number of spans. M is the bending moment of the external forces for any section whose abscissa is x, and whose moment of inertia with respect to the neutral axis is I. Unless otherwise stated I will be regarded as constant, that is, the cross-section of the beam is constant throughout its length.

To obtain the particular equation of the elastic curve for any special case, it is first necessary to express M as a function of x and then integrate the general equation twice. The ordinate y will then be known for any value of x. It should however be borne in mind that formula (5), like formula (4), is only true when the unit-stress S is less than the elastic limit of the material.

Prob. 61. A wooden beam ½ inch wide, ¾ inch deep, and 3 feet span carries a load of 14 pounds at the middle. Find the radius of curvature for the middle, quarter points and ends.

ART. 34. DEFLECTION OF CANTILEVERS.

Case I. A load at the free end.—Take the origin of co-ordinates at the free end, and as in Fig. 22, let m be any point of the elastic curve whose abscissa is x and ordinate y. For this point the bending moment M is $-Px$ and the general formula (5) becomes,

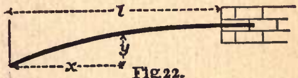


Fig. 22.

$$EI\frac{d^2y}{dx^2} = -Px.$$

By integration,

$$EI\frac{dy}{dx} = -\frac{Px^2}{2} + C.$$

But $\frac{dy}{dx}$ is the tangent of the angle which the tangent to the elastic curve at m makes with the axis of x and as the beam is fixed at the wall the value of $\frac{dy}{dx}$ is 0 when x equals l. Hence

$C = \frac{1}{2}Pl^2$, and the first differential equation is,

$$EI\frac{dy}{dx} = \frac{Pl^2}{2} - \frac{Px^2}{2}.$$

The second integration now gives,

$$EIy = \frac{Pl^2x}{2} - \frac{Px^3}{6} + C'.$$

But $y = 0$, when $x = 0$. Hence $C' = 0$, and

$$6EIy = P(3l^2x - x^3),$$

which is the equation of the elastic curve for a cantilever of length l with a load P at the free end. If $x = l$ the value of y will be the maximum deflection, which may be represented by Δ. Then,

$$\Delta = \frac{Pl^3}{3EI},$$

and for any point of the beam the deflection is $\Delta - y$.

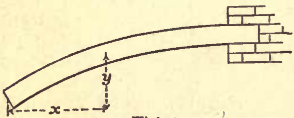


Fig. 23.

Case II. A cantilever uniformly loaded.—Let the origin be taken at the free end as before. Let the load per linear unit be w. Then for any section $M = -\frac{1}{2}wx^2$ and formula (5) becomes,

$$EI\frac{d^2y}{dx^2} = \frac{wx^2}{2}.$$

Integrate this, determine the constant of integration by the consideration that $\frac{dy}{dx} = 0$ when $x = l$, and then,

$$6EI\frac{dy}{dx} = wl^3 - wx^3.$$

Intergrate again, and after determining the constant, the equation of the elastic curve is,

$$24EIy = w(4l^3x - x^4),$$

which is a biquadratic parabola. For $x = l$, $y = \Delta$ the maximum deflection, whose value is,

$$\Delta = \frac{wl^4}{8EI} = \frac{Wl^3}{8EI},$$

where W is the total uniform load on the cantilever.

Case III. A cantilever uniformly loaded with W and also carrying a load P at the free end.—Here it is easy to show that the maximum deflection is,

$$\Delta = \frac{3Wl^3 + 8Pl^3}{24EI}.$$

In all these cases the maximum unit-stress S produced by the loads must not exceed the elastic limit of the material.

Prob. 62. Compute the deflection of a wooden cantilever, 2 × 2 inches and 6 feet span, caused by a load of 100 pounds at the end. Also of a cast iron cantilever of the same dimensions.

Prob. 63. In order to find the coefficient of elasticity of a cast iron bar 2 inches wide, 4 inches deep and 6 feet long, it was balanced upon a support and a weight of 4 000 pounds hung at each end, when the deflection of the ends was observed to be 0.401 inches. Compute the value of E.

ART. 35. DEFLECTION OF SIMPLE BEAMS.

Case I. A single load P at the middle.—Let the origin be taken at the left support. For any section between the left support and the middle the bending moment M is $\frac{1}{2}Px$. Then the general formula (5) becomes,

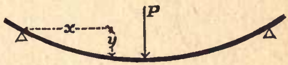


Fig. 24.

$$EI\frac{d^2y}{dx^2} = \frac{Px}{2}.$$

Integrate this and find the constant by the fact that $\frac{dy}{dx} = 0$ when $x = \frac{1}{2}$. Then integrate again and find the constant by the fact that $y = 0$ when $x = 0$. Thus,

$$48EIy = P(4x^3 - 3l^2x),$$

is the equation of elastic curve between the left hand support and the load. For the greatest deflection make $x = \frac{1}{2}l$, then,

$$\Delta = \frac{Pl^3}{48EI}.$$

This result may also be obtained by regarding the beam as a cantilever fixed at the load and bent upward by the reaction.

Case II. A uniform load.—Let w be the load per linear unit, then the formula (5) becomes,

$$EI\frac{d^2y}{dx^2} = \frac{wlx}{2} - \frac{wx^2}{2}.$$

Integrate this twice, find the constants as in the preceding paragraph, and the equation of the elastic curve is,

$$24EIy = w(-x^4 + 2lx^3 - l^3x),$$

from which the maximum deflection is,

$$\Delta = \frac{5wl^4}{384EI} = \frac{5Wl^3}{384EI}.$$

Case III. A load P at any point.—Here it is necessary first to consider that there are two elastic curves, one on each side of the load, which have distinct equations, but which have a common tangent and ordinate under the load. As in

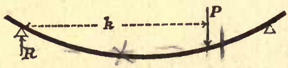


Fig. 25.

Fig. 25, let the load be placed at a distance k from the left support. Then the left reaction is $P\frac{l-k}{l}$. From the general formula (5), with the origin at the left support, the equations are,

On the left of the load,

$$(a)\quad EI\frac{d^2y}{dx^2} = Rx,$$

$$(b)\quad EI\frac{dy}{dy} = \tfrac{1}{2}Rx^2 + C_1,$$

$$(c)\quad EIy = \tfrac{1}{6}Rx^3 + C_1x + C_2.$$

On the right of the load,

$$(a)'\quad EI\frac{d^2y}{dx^2} = Rx - P(x-k),$$

$$(b)'\quad EI\frac{dy}{dx} = \tfrac{1}{2}Rx^2 - \tfrac{1}{2}Px^2 + Pkx + C_2,$$

$$(c)'\quad EIy = \tfrac{1}{6}Rx^3 - \tfrac{1}{6}Px^3 + \tfrac{1}{2}Pkx^2 + C_2x + C_4,$$

To determine the constants consider in (c) that $y = 0$ when $x = 0$ and hence that $C_3 = 0$. Also in $(c)'$, $y = 0$ when $x = l$ and hence C_4 is known in terms of R, P and C_2. Since the curves have a common tangent under the load, $(b) = (b)'$ when $x = k$, and since they have a common ordinate at that point $(c) = (c)'$ when $x = k$. Or,

$$\tfrac{1}{2}Rk^2 + C_1 = \tfrac{1}{2}Rk^2 - \tfrac{1}{2}Pk^2 + Pk^2 + C_2,$$

$$\tfrac{1}{6}Rk^3 + C_1k = \tfrac{1}{6}R(k^3 - l^3) - \tfrac{1}{6}P(k^3 - l^3) + \tfrac{1}{2}Pk(k^2 - l^2) + C_2(k - l).$$

From these two equations the values of C_1 and C_2 are found. Then the equation of the elastic curve on the left of the load is,

$$6EIy = Rx^3 + Pk\left(3k - 2l - \frac{k^2}{2}\right)x.$$

To find the maximum deflection, insert in this the value of R and find the value of x for which $\frac{dy}{dx}$ becomes 0. If k be greater than $\frac{1}{2}l$ this value of x inserted in the above equation gives the maximum value of y. If k be less than $\frac{1}{2}l$ the maximum deflection is on the other side of the load. For instance, if $k = \frac{3}{4}l$, the equation of the elastic curve on the left of the load is,

$$384EIy = 16Px^2 - 15Pl^2x.$$

This is a maximum when $x = 0.56l$, which is the point of greatest deflection.

Prob. 64. In order to find the coefficient of elasticity of Quercus alba a bar 4 centimeters square and one meter long was supported at the ends and loaded in the middle with weights of 50 and 100 kilograms when the corresponding deflections were found to be 6.6 and 13.0 millimeters. Show that the mean value of E was 74 500 kilos per square centimeter.

ART. 36. COMPARATIVE DEFLECTION AND STIFFNESS.

From the two preceding articles the following values of the maximum deflections may now be written and their comparison will show the relative stiffness of the different cases.

For a cantilever loaded at the end with W, $\Delta = \frac{1}{3} \cdot \frac{Wl^3}{EI}$.

For a cantilever uniformly loaded with W, $\Delta = \frac{1}{8} \cdot \frac{Wl^3}{EI}$.

For a simple beam loaded at middle with W, $\Delta = \frac{1}{48} \cdot \frac{Wl^3}{EI}$.

For a simple beam uniformly loaded with W, $\Delta = \frac{5}{384} \cdot \frac{Wl^3}{EI}$.

The relative deflections of these four cases are hence as the numbers 1, $\frac{3}{8}$, $\frac{1}{16}$ and $\frac{5}{128}$.

These equations also show that the deflections vary directly as the load, directly as the cube of the length and inversely as E and I. For a rectangular beam $I = \frac{bd^3}{12}$, and hence the deflection of a rectangular beam is inversely as its breadth and inversely as the cube of its depth.

The stiffness of a beam is indicated by the load that it can carry with a given deflection. From the above it is seen that the value of the load is,

$$W = \frac{mEI\Delta}{l^3},$$

where m has the value 3, 8, 48, or $\frac{384}{5}$ as the case may be. Therefore, the stiffness of a beam varies directly as E, directly as I and inversely as the cube of its length, and the relative stiffness of the above four cases is as the numbers 1, $2\frac{2}{3}$, 16 and $25\frac{3}{5}$. From this it appears that the laws of stiffness are very different from those of strength. (Art. 28.)

Prob. 65. Compare the strength and stiffness of a joint 3×8 inches when laid with flat side vertical and when laid with narrow side vertical.

Prob. 66. Find the thickness of a white pine plank of 8 feet span required not to bend more than $\frac{1}{480}$th of its length under a head of water of 20 feet.

ART. 37. RELATION BETWEEN DEFLECTION AND STRESS.

Let the four cases discussed in Arts. 28 and 36 be again considered. For the strength,

$$W = n\frac{SI}{lc}, \quad \text{where } n = 1, 2, 4 \text{ or } 8.$$

For the stiffness,

$$W = m\frac{EI\Delta}{l^3}, \quad \text{where } m = 3, 8, 48 \text{ or } 76\tfrac{4}{5}.$$

By equating these values of W the relation between Δ and S is obtained, thus,

$$S = \frac{mEc\Delta}{nl^2}, \qquad \text{or} \qquad \Delta = \frac{nl^2S}{mcE}.$$

These equations, like the general formula (4) and (5), are only valid when S is less than the elastic limit of the materials.

This also shows that the maximum deflection Δ varies as $\frac{l^2}{c}$ for beams of the same material under the same unit-stress S.

Prob. 67. Find the deflection of a wrought iron I heavy 10 inch beam of 9 feet span when strained by a uniform load up to the elastic limit.

Prob. 68. Compare the deflections of a wrought iron and wooden beam when strained to the elastic limit by a single load at the middle.

ART. 38. RECAPITULATION.

Let the length of the cantilever or beam be l, the load upon it, whether concentrated or uniform, be W, the moment of inertia of the constant cross-section about a horizontal axis through its center of gravity be I, the shortest distance from the remotest fiber to said axis be c, the unit-stress at the elastic limit be S_e, and the coefficient of elasticity be E. Then, from the preceding articles, the following table may be compiled, which exhibits the most important results relating to both absolute and relative strength and stiffness.

Case.	Max. Vertical Shear.	Max. Bending Moment.	Max. Allow'ble Load.	Max. Deflection.	Relative Strength.	Relative Stiffness.
Cantilever loaded at end	W	Wl	$\frac{S_eI}{lc}$	$\frac{1}{3}\frac{Wl^3}{EI}$	1	1
Cantilever loaded uniformly	W	$\frac{1}{2}Wl$	$2\frac{S_eI}{lc}$	$\frac{1}{8}\frac{Wl^3}{EI}$	2	$2\frac{2}{3}$
Simple beam loaded at middle	$\frac{1}{2}W$	$\frac{1}{4}Wl$	$4\frac{S_eI}{lc}$	$\frac{1}{48}\frac{Wl^3}{EI}$	4	16
Simple beam loaded uniformly	$\frac{1}{2}W$	$\frac{1}{8}Wl$	$8\frac{S_eI}{lc}$	$\frac{5}{384}\frac{Wl^3}{EI}$	8	$25\frac{3}{5}$

Here the signs of the maximum shears and moments are omitted as only their absolute values are needed in computations. Evidently the moments are negative for the first and second cases, and positive for the third and fourth, the direction of the curvature being different.

Prob. 69. A wooden beam of breadth b, depth d and span x is loaded with P at the middle. Find the value of x so that rupture may occur under the load. Find also the value of x so that rupture may occur by shearing at the supports.

Art. 39. Cantilevers of Constant Strength.

All cases thus far discussed have been of constant cross-section throughout their entire length. But in the general formula (4) the unit-stress S is proportional to the bending moment M, and hence varies throughout the beam in the same way as the moments vary. Hence some parts of the beam are but slightly strained in comparison with the dangerous section.

A beam of uniform strength is one so shaped that the unit-stress S is the same in all fibers at the upper and lower surfaces. Hence to ascertain the form of such a beam the unit-stress S in

(4) must be taken as constant and $\frac{I}{c}$ be made to vary with M. The discussion will be given only for the most important practical cases, namely those where the sections are rectangular. For these $\frac{I}{c}$ equals $\frac{bd^2}{6}$, and formula (4) becomes,

$$\frac{Sbd^2}{6} = M.$$

In this bd^2 must vary with M for forms of uniform strength.

For a cantilever with a load P at the end, $M = Px$ and the equation becomes $\frac{1}{6}Sbd^2 = Px$, in which P and S are constant. If the breadth be taken as constant, d^2 varies with x and the profile is that of a parabola whose vertex is at the load, as shown in Fig. 26. The equation of the parabola is $d^2 = \frac{6P}{Sb}x$ from which d may be found for given values of x. The walking beam of an engine is often made approximately of this shape.

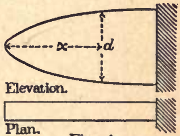


Fig. 26

If the depth of the cantilever be constant then b varies directly as x and hence the plan of the cantilever is a triangle as shown in Fig. 27. The value of b may be found from the expression $b = \frac{6Px}{Sd^2}$.

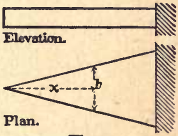


Fig. 27

For a cantilever uniformly loaded with w per linear unit $M = \frac{1}{2}wx^2$, and the equation becomes $\frac{1}{6}Sbd^2 = \frac{1}{2}wx^2$, in which w and S are known. If the breadth be taken as constant then d varies as x and the elevation is a triangle, as in Fig. 28, whose depth at any point is $d = x\sqrt{\frac{3w}{Sb}}$. If however the depth be taken constant, then $b = \frac{3w}{Sd^2}x^2$ which is the equa-

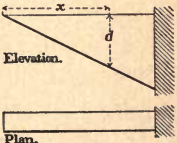


Fig. 28.

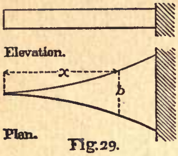


Fig. 29.

tion of a parabola whose vertex is at the free end of the cantilever and whose axis is perpendicular to it. Or the equation may be satisfied by two parabolas as shown in Fig. 29.

The vertical shear modifies in practice the shape of these forms near their ends. For instance, a cantilever loaded at the end with P requires a cross-section at the end equal to $\frac{P}{S_c}$ where S_c is the working shearing strength. This cross-section must be preserved until a value of x is reached, where the same value of the cross-section is found from the moment.

The deflection of a cantilever of uniform strength is evidently greater than that of one of constant cross-section, since the unit-stress S is greater throughout. In any case it may be determined from the general formula (5) by substituting for M and I their values in terms of x, integrating twice, determining the constants, and then making x equal to l for the maximum value of y.

For a cantilever loaded at the end and of constant breadth, as in Fig. 26, formula (5) becomes,

$$\frac{d^2y}{dx^2} = \frac{12Px}{Ebd^3} = \frac{2}{E}\sqrt{\frac{S^3b}{6Px}}.$$

Integrating this twice and determining the constants, as in Art. 34, the equation of the elastic curve is found to be,

$$y = \frac{2}{E}\sqrt{\frac{S^3b}{6P}}\left(\frac{4}{3}x^{\frac{3}{2}} - 2l^{\frac{1}{2}}x\right)$$

In this make $x = l$, and substitute for S its value $\frac{6Pl}{bd_1^2}$, where d_1 is the depth of the wall. Then,

$$\Delta = \frac{8Pl^3}{Ebd_1^3},$$

which is double that of a cantilever of constant cross-section.

For a cantilever loaded at the end and of constant depth, formula (5) becomes,

$$\frac{d^2y}{dx^2} = \frac{12Px}{Ebd^3} = \frac{2S}{Ed}$$

By intergrating this twice and determining the constants as before, the equation of the elastic curve is found, from which the deflection is,

$$\Delta = \frac{6Pl^3}{Ebd^3},$$

which is fifty per cent greater than for one of uniform section.

Prob. 70. A cast-iron cantilever of uniform strength is to be 4 feet long, 3 inches in breadth and to carry a load of 15 000 pounds at the end. Find the proper depths for every foot in length, using 3 000 pounds per square inch for the horizontal unit-stress, and 4 000 pounds per square inch for the shearing unit-stress.

ART. 40. SIMPLE BEAMS OF UNIFORM STRENGTH.

In the same manner it is easy to deduce the forms of uniform strength for simple beams of rectangular cross-section.

For a load at the middle and breadth constant, $M = \frac{1}{2}Px$, and hence, $\frac{1}{6}Sbd^2 = \frac{1}{2}Px$. Hence $d^2 = \frac{3P}{Sb}x$, from which values of d may be found for assumed values of x. Here the profile of the beam will be parabolic, the vertex being at the support, and the maximum depth under the load.

For a load at the middle and depth constant, $M = \frac{1}{2}Px$ as before. Hence $b = \frac{3P}{Sd^2}x$, and the plan must be triangular or lozenge shaped, the width uniformly increasing from the support to the load.

For a uniform load and constant breadth, $M = \frac{1}{2}wlx - \frac{1}{2}wx^2$, and hence, $d^2 = \frac{3w}{Sb}(lx - x^2)$, and the profile of the beam must be elliptical, or preferably a half-ellipse, For a uniform load and constant depth, $b = \frac{3w}{Sd^2}(lx - x^2)$ and hence the plan should be formed of two parabolas having their vertices at the middle of the span.

The figures for these four cases are purposely omitted, in order that the student may draw them on the margin. In the same manner as in the last Article, it can be shown that the deflection of a beam of uniform strength loaded at the middle is double that of one of constant cross-section if the breadth is constant, and is one and one-half times as much if the depth is constant.

Prob. 71. Find the deflection of a steel spring of constant depth and uniform strength which is 6 inches wide at the middle, 52 inches long, and loaded at the middle with 600 pounds, the depth being such that this load strains the material to one-half its elastic limit.

CHAPTER IV.

ON RESTRAINED BEAMS AND ON CONTINUOUS BEAMS.

ART. 41. GENERAL PRINCIPLES.

A restrained beam is one whose ends are fastened in walls, or so arranged that the tangents to the elastic curve at the ends always remain horizontal. The simplest case of a restrained beam is a cantilever with one end fixed and the other free. Two other common cases are a beam fixed at one end and supported at the other, and a beam fixed at both ends.

A continuous beam is one supported upon several points in the same horizontal plane. A simple beam may be regarded as a particular case of a continuous beam where the number of supports is two. The ends of a continuous beam are said to be free when they overhang, supported when they merely rest on abutments, and restrained when they are horizontally fixed in walls.

The general principles of the preceding Chapter hold good for all kinds of beams. If a plane be imagined to cut any beam at any point the laws of Arts. 15 and 18 apply to the stresses in that section. The resisting shear and the resisting moment for that section have the values deduced in Art. 19, and as in Art. 20 the two fundamental formulas for investigation are,

$$(3) \qquad S_sA = V,$$

$$(4) \qquad \frac{SI}{c} = M.$$

Here S_s is the vertical shearing unit-stress in the section, and S is the horizontal tensile or compressive unit-stress on the fiber most remote from the neutral axis; c is the shortest distance from that fiber to that axis; I the moment of inertia, and A the area of the cross-section. V is the vertical shear of the external forces on the left of the section, and M is the bending moment of those forces with reference to a point in the section. For any given beam evidently S_s and S may be found for any section as soon as V and M are known.

The general equation of the elastic line, deduced in Art. 33, is also valid for all kinds of beams. It is,

$$(5) \qquad \frac{d^2y}{dx^2} = \frac{M}{EI}$$

where x is the abscissa and y the ordinate of any point of the elastic curve, M being the bending moment for that section, and E the coefficient of elasticity of the material.

The vertical shear V is the algebraic sum of the external forces on the left of the section, or, as in Art. 16,

$V =$ Reactions on left of section minus loads on left of section.

For simple beams and cantilevers the determination of V for any special case was easy, as the left reaction could be readily found for any given loads. For restrained and continuous beams, however, it is not, in general, easy to find the reactions, and hence a different method of determining V is necessary. Let Fig. 30 represent one span of a continuous or restrained beam. Let V be the vertical shear for any section at the distance x from the left support, and V' the vertical shear at a section infinitely near to the left support. Also let ΣP denote the sum of all the concentrated loads on the distance x, and wx the uniform load,

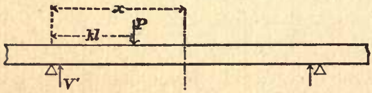


Fig 30.

Then because V' is the algebraic sum of all the vertical forces on its left, the definition of vertical shear gives,

$$(6) \qquad V = V' - \Sigma P - wx.$$

Hence V can be determined as soon as V' is known.

The bending moment M is the algebraic sum of the moments of the external forces on the left of the section with reference to a point in that section, or, as in Art. 17,

$M =$ Moments of reactions minus moments of loads.

For the reason just mentioned it is in general necessary to determine M for continuous and restrained beams by a different method. Let M' denote the bending moment at the left support of any span as in Fig. 30, and M'' that at the right support, while M is the bending moment for any section distant x from the left support. Let P be any concentrated load upon the space x at a distance kl from the left support, k being a fraction less than unity, and let w be the uniform load per linear unit. Then, because M' is the algebraic sum of all the moments of the external forces on its left, the definition of bending moment gives,

$$(7) \qquad M = M' + V'x - \Sigma P(x - kl) - \tfrac{1}{2}wx^2.$$

Hence M may be found for any section as soon as V' and M' are known.

The relation between the bending moment and the vertical shear at any section is interesting and important. At the section x the moment is M and the shear is V. At the next consecutive section $x + dx$ the moment is $M + dM$, which may also be expressed by $M + Vdx$. Hence,

$$V = \frac{dM}{dx}.$$

This may be proved otherwise by differentiating (7) and comparing with (6).

The vertical shear V' at the support may be easily found if the bending moments M' and M'' be known. Thus in equation

(7) make $x = l$, then M becomes M'', and hence,

$$(8) \qquad V'' = \frac{M'' - M'}{l} + \frac{wl}{2} + \Sigma P(l - k).$$

The whole problem of the discussion of restrained and continuous beams hence consists in the determination of the bending moments at the supports. When these are known the values of M and V may be determined for every section, and the general formulas (3), (4) and (5) be applied as in Chapter III, to the investigation of questions of strength and deflection. The formulas (6), (7) and (8) apply to simple beams and cantilevers also. For simple beams $M' = M'' = 0$, and $V' = R$ since there is no restraint at the ends. For cantilevers $M' = 0$ for the free end, and M'' is the moment at the wall.

Prob. 72. A bar of length $2l$ and weighing w per linear unit is supported at the middle. Apply formulas (6) and (7) to the statement of general expressions for the moment and shear at any section on the left of the support, and also at any section on the right of the support.

Art. 42. Beams Overhanging One Support.

A cantilever has its upper fibers in tension and the lower in compression, while a simple beam has its upper fibers in compression and the lower in tension. Evidently a beam overhanging one support, as in Fig. 31, has its overhanging part in the condition of a cantilever, and the part near the other end in the condition of a simple beam. Hence there must be a point i where the stresses

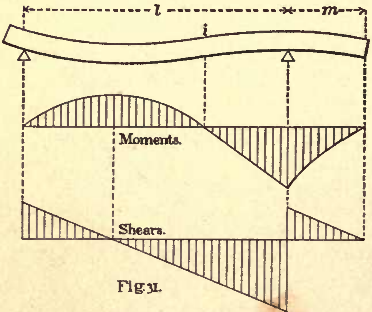


Fig. 31.

change from tension to compression, and where the curvature changes from positive to negative. This point i is called the inflection point; it is the point where the bending moment is zero. Since the beam has but two supports its reactions may be found, as in Art. 14, and the entire investigation be made by the principles of the last Chapter.

Consider a beam loaded uniformly with w per linear unit. Let l be the distance between the supports and m the length of the overhanging portion. Let the left reaction be R_1 and the right R_2. Then, for any section distant x from the left support,

When x is less than l,	When x is greater than l,
$V = R_1 - wx,$	$V = R_1 + R_2 - wx,$
$M = R_1x - \frac{1}{2}wx^2.$	$M = R_1x + R_2(x-l) - wx^2.$

The curves corresponding to these equations are shown on Fig. 31. The shear curve consists of two straight lines; $V = R_1$ when $x = 0$, $V = 0$ when $x = \frac{R_1}{w}$; at the right support $V = R_1 - wl$ from the first equation, and $V = R_1 + R_2 - wl$ from the second; $V = 0$ when $x = l + m$. The moment curve consists of two parts of parabolas; $M = 0$ when $x = 0$, M is a maximum when $x = \frac{R_1}{w}$, $M = 0$ at the inflection point where $x = \frac{2R_1}{w}$, M has its negative maximum when $x = l$, and $M = 0$ when $x = l + m$. The diagrams show clearly the distribution of shears and moments throughout the beam.

For example, if $l = 20$ and $m = 10$ the reactions are found to be $R_1 = 7.5w$ and $R_2 = 22.5w$. Then the point of zero shear or maximum moment is at $x = 7.5$, the inflection point at $x = 15$, the maximum shears are $+ 7.5w$, $- 12.5w$ and $+ 10w$, and the maximum moments are $+ 56.25w$ and $- 50w$. The relative values of the two maximum moments depend on the ratio of m to l. If this ratio be zero there is no overhanging part and no negative moment, and the beam is a simple one. If $m = \frac{1}{2}l$ the case is that just discussed. If $m = l$ the beam becomes a cantilever supported at the middle.

After having thus found the maximum values of V and M the beam may be investigated by the application of formulas (3) and (4) in the same manner as a cantilever or simple beam. By the use of formula (5) the equation of the elastic curve between the two supports is found to be,

$$24EIy = 4R_1(x^3 - l^2x) - w(x^4 - l^3x).$$

From this the maximum deflection for any particular case may be determined by putting $\frac{dy}{dx}$ equal to zero, solving for x, and then finding the corresponding value of y.

Prob. 73. Find the ratio of m to l so that the maximum positive moment may equal numerically the maximum negative moment.

Prob. 74. A light 12-inch I beam 25 feet long is used as a floor beam in a bridge with one sidewalk, the distance between the supports being 20 feet. Find its factor of safety when the whole beam is loaded with 1 200 pounds per linear foot, and also when only the 20 feet roadway is loaded.

ART. 43. BEAMS FIXED AT ONE END AND SUPPORTED AT THE OTHER.

When a beam is fixed horizontally in a wall at one end while the other end is merely supported, the restraint at the fixed end causes the distribution of moments to be similar to that of a beam with one overhanging end. The reaction at the supported end cannot be found by the principles of statics as in Art. 14, but must be determined by the help of the equation of the elastic curve.

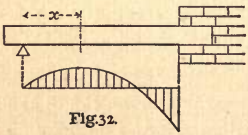


Fig.32.

Case I.—For a uniform load over the whole beam let R be the left reaction as in Fig. 32. Then for any section the bending moment is $Rx - \frac{1}{2}wx^2$. Hence the differential equation of the elastic curve is,

$$EI\frac{d^2y}{dx^2} = Rx - \tfrac{1}{2}wx^2.$$

Integrate this once and determine the constant from the necessary condition that $\frac{dy}{dx} = 0$ when $x = l$. Integrate again and find the constant from the fact that $y = 0$ when $x = 0$. Then,

$$24EIy = 4R(x^3 - 3l^2x) - w(x^4 - 24l^3x).$$

Here also $y = 0$ when $x = l$, and therefore $R = \frac{3}{8}wl$.

The moment at any point now is $M = \frac{3}{8}wlx - \frac{1}{2}wx^2$, and by placing this equal to zero it is seen that the point of inflection is at $x = \frac{3}{4}l$. By the method of Art. 23 it is found that the maximum moments are $+\frac{9}{128}wl^2$ and $-\frac{1}{8}wl^2$, and that the distribution of moments is as represented in Fig. 3.2

The point of maximum deflection is found by placing $\frac{dy}{dx}$ equal to zero and solving for x. This gives $8x^3 - 9lx^2 + l^3 = 0$, one root of which is $x = +0.4215l$, and this inserted in the value of y gives,

$$\Delta = 0.0054\,\frac{wl^3}{EI},$$

for the value of the maximum deflection.

Case II.—For a load at the middle it is first necessary to consider that there are two elastic curves having a common ordinate and a common tangent under the load, since the expressions for the moment are different on opposite sides of the load. Thus taking the origin as usual at the supported end,

On the left of the load,

(*a*) $$EI\frac{d^2y}{dx^2} = Rx,$$

(*b*) $$EI\frac{dy}{dx} = \tfrac{1}{2}Rx^2 + C,$$

(*c*) $$EIy = \tfrac{1}{6}Rx^3 + C_1x + C_3.$$

On the right of the load the similar equations are,

$$(a)' \qquad EI\frac{d^2y}{dx^2} = Rx - P(x - \tfrac{1}{2}l),$$

$$(b)' \qquad EI\frac{dy}{dx} = \tfrac{1}{2}Rx^2 - \tfrac{1}{2}Px^2 + \tfrac{1}{2}Plx + C_2,$$

$$(c)' \qquad EIy = \tfrac{1}{6}Rx^3 - \tfrac{1}{6}Px^3 + \tfrac{1}{4}Plx^2 + C_2x + C_4.$$

To determine the constants consider in (c) that $y = 0$ when $x = 0$ and hence that $C_3 = 0$. In $(b)'$ the tangent $\frac{dy}{dx} = 0$ when $x = l$ and hence $C_2 = -\tfrac{1}{2}Rl$. Since the curves have a common tangen under the load $(b) = (b)'$ for $x = \tfrac{1}{2}l$, and thus the value of C_1 is found. Since the curves have a common ordinate under the load $(c) = (c)'$ when $x = \tfrac{1}{2}l$, and thus C_4 is found. Then,

$$(c) \qquad EIy = \frac{Rx^3}{6} + \frac{Pl^2x}{8} - \frac{Rl^2x}{2},$$

$$(c)' \qquad EIy = \frac{Rx^3}{6} - \frac{Px^3}{6} + \frac{Plx^2}{4} - \frac{Rl^2x}{2} + \frac{Pl^3}{48}.$$

From the second of these the value of the reaction is $R = \frac{5}{16}P$.

The moment on the left of the load is now $M = \frac{5}{16}Px$, and that on the right $M = -\frac{11}{16}Px + \tfrac{1}{2}Pl$. The maximum positive moment obtains at the load and its value is $\frac{5}{32}Pl$. The maximum negative moment occurs at the wall, and its value is $\frac{3}{16}Pl$. The inflection point is at $x = \frac{8}{11}l$. The deflection under the load is readily found from (c) by making $x = \frac{1}{2}l$. The maximum deflection occurs at a less value of x, which may be found by putting $\frac{dy}{dx} = 0$.

Case III. For a load at any point whose distance from the left support is kl, the following results may be deduced by a method exactly similar to that of the last case.

Reaction at supported end	=	$\tfrac{1}{2}P(2 - 3k + k^3)$.
Reaction at fixed end	=	$\tfrac{1}{2}P(3k - k^3)$.
Maximum positive moment	=	$\tfrac{1}{2}Plk(2 - 3k^3 + k^3)$.
Maximum negative moment	=	$\tfrac{1}{2}Pl(k - k^3)$.

The absolute maximum deflection occurs under the load when $k = 0.422l$.

Prob. 75. Find the position of load P which gives the maximum positive moment. Find also the position which gives the maximum negative moment.

ART. 44. BEAMS OVERHANGING BOTH SUPPORTS.

When a beam overhangs both supports, as in Fig. 33, there will be a negative moment at each support and, in general, two inflection points and a positive moment. The relative values of these will depend upon the lengths of the over-

Fig. 33.

hanging parts. If these lengths m and n be equal, the reactions and negative moments will be equal and the two halves of the beam be symmetrically strained by a uniform load. In any case, whatever be the nature of the load, the reactions may be found by Art. 14 and the maximum vertical shears and bending moments be determined by the methods of the last Chapter.

Prob. 76. If $m = n$ in Fig. 33, find the ratio of m to l in order that the maximum positive moment may numerically equal the maximum negative moments.

Prob. 77. A bridge with two sidewalks has a wooden floor beam 14×15 inches and 30 feet long, the distance between supports being 20 feet and each sidewalk 5 feet. Find its factor of safety under a uniformly distributed load of 29 000 pounds.

ART. 45. BEAMS FIXED AT BOTH ENDS.

Case I. For a uniform load it is evident that the bending moments at the supports are equal. Then, for any section x, formula (7) becomes,

$$M = M' + Rx - \tfrac{1}{2}wx^2,$$

in which M' is the unknown moment at the left support. By inserting this in (5) and integrating twice, making $\frac{dy}{dx} = 0$ when $x = 0$ and also when $x = l$, the value of M' is found to be $-\frac{wl^2}{12}$, and the equation of the elastic curve is,

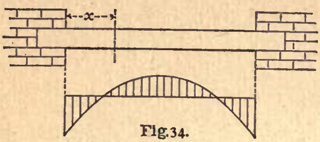


Fig. 34.

$$24EIy = w(-l^2x^2 + 2lx^3 - x^4).$$

From this the maximum deflection is found to be,

$$\Delta = \frac{wl^4}{384EI}.$$

The inflection points are located by making $M = 0$. This gives $x = \frac{1}{2}l \pm l\sqrt{\frac{1}{12}}$. The maximum positive moment is at the middle and its value is $\frac{wl^2}{24}$.

Case II. For a load at the middle, the bending moment between the left end and the load is,

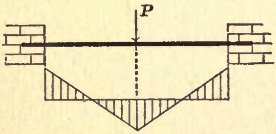


Fig. 35.

$$M = M' + \frac{1}{2}Px,$$

and in a similar manner to that of the last case it is easy to find that the maximum negative moments are $\frac{1}{8}Pl$, that the maximum positive moments are $\frac{1}{8}Pl$, and that the maximum deflection is $\frac{Pl^3}{192EI}$.

Case III. For a load P at a distance kl from the left end, a method similar to that of Case II, Art. 43, may be followed. Let M' and R' denote the unknown moment and reaction at the left end and M'' and R'' those for the right end. Then,

On the left of the load,

$$(a) \qquad EI\frac{d^2y}{dx^2} = M' + R'x,$$

$$(b) \qquad EI\frac{dy}{dx} = M'x + \frac{1}{2}R'x^2,$$

$$(c) \qquad EIy = \frac{1}{2}M'x^2 + \frac{1}{6}R'x^2,$$

On the right of the load,

$$(a)' \quad EI\frac{d^2y}{dx^2} = M' + R'x + P(x - kl)$$

$$(b)' \quad EI\frac{dy}{dx} = M'x + \tfrac{1}{2}R'x^2 + \tfrac{1}{2}Px^2 - \tfrac{1}{2}Pkl + C_1,$$

$$(c)' \quad EIy = M'x + \tfrac{1}{2}R'x^2 + \tfrac{1}{2}Px^2 - \tfrac{1}{2}Pkl + C_1 + C_2.$$

The constants of integration are here readily determined since both $\frac{dy}{dx}$ and y become zero for $x = 0$ and for $x = l$. Then if $x = kl$, the tangent (b) is equal to $(b)'$, and the ordinate (c) to $(c)'$. This gives two equations from which the values of M' and R' are found. Thus the following results are deduced,

Reaction at left end	$= P(1 - 3k^2 + 2k^3)$,
Reaction at right end	$= Pk^2(3 - 2k)$,
Moment at left end	$= -Plk(1 - 2k + k^2)$,
Moment at right end	$= -Plk^2(1 - k)$,
Moment under load	$= +Plk^2(2 - 4k + 2k^2)$.

If $k = \frac{1}{2}$ the load is at the middle and these results reduce to the values found in Case II.

Prob. 78. What wrought iron I beam is required for a span of 24 feet to support a uniform load of 40 000 pounds, the ends being merely supported? What one is needed when the ends are fixed?

ART. 46. COMPARISON OF RESTRAINED AND SIMPLE BEAMS.

As the maximum moments for restrained beams are less than for simple beams their strength is relatively greater. This was to be expected since the restraint produces a negative bending moment and lessens the deflection which would otherwise occur. The comparative strength and stiffness of cantilevers and simple beams is given in Art. 38. To these may now be added four cases from Arts. 43 and 45, and the following table be formed, in which W represents the total load, whether uniform or concentrated.

Beams of Uniform Cross-section.	Maximum Moment.	Maximum Deflection.	Relative Strength.	Relative Stiffness.
Cantilever, load at end	Wl	$\frac{1}{3}\frac{Wl^3}{EI}$	1	1
Cantilever, uniform load	$\frac{1}{2}Wl$	$\frac{1}{8}\frac{Wl^3}{EI}$	2	$2\frac{2}{3}$
Simple beam, load at middle	$\frac{1}{4}Wl$	$\frac{1}{48}\frac{Wl^3}{EI}$	4	16
Simple beam uniformly loaded	$\frac{1}{8}Wl$	$\frac{5}{384}\frac{Wl^3}{EI}$	8	$25\frac{3}{5}$
Beam fixed at one end, supported at other, load near middle	$0.174Wl$	$0.099\frac{Wl^3}{EI}$	6	34
Beam fixed at one end, supported at other, uniform load	$\frac{1}{8}Wl$	$0.0054\frac{Wl^3}{EI}$	8	62
Beam fixed at both ends, load at middle	$\frac{1}{8}Wl$	$\frac{1}{192}\frac{Wl^3}{EI}$	8	64
Beam fixed at both ends, uniform load	$\frac{1}{12}Wl$	$\frac{1}{384}\frac{Wl^3}{EI}$	12	128

This table shows that a beam fixed at both ends and uniformly loaded is one and one-half times as strong and five times as stiff as a simple beam under the same load. The advantage of fixing the ends is hence very great.

Prob. 79. Find the deflection of a **I** 9-inch beam of 6 feet span and fixed ends when strained so that the tensile and compressive stresses at the dangerous section are 14 000 pounds per square inch.

Art. 47. Properties of Continuous Beams.

The theory of continuous beams presented in the following pages includes only those with constant cross-section having the supports on the same level, as only such are used in engineering constructions. Unless otherwise stated, the ends will be supposed to simply rest upon their supports, so that there can be no moments at those points. Evidently then the end spans are somewhat in the condition of a beam with one overhanging end and the other spans somewhat in the condition of a beam with two overhanging ends. At each intermediate support there is a negative moment, and the distribution of moments throughout the beam will be as represented in Fig. 36.

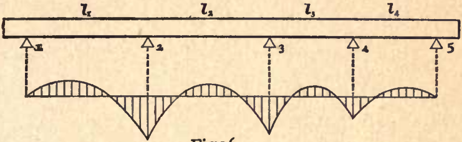


Fig. 36.

As shown in Art. 41 the investigation of a continuous beam depends upon the determination of the bending moments at the supports. In the case of Fig. 36 these moments being those at the supports 2, 3 and 4, may be designated M_2, M_3 and M_4. Let V_1, V_2, V_3 and V_4 denote the vertical shear at the right of each support. The first step is to find the moments M_2, M_3 and M_4. Then from formula (8) the values of V_1, V_2, V_3 and V_4 are found, and thus by formula (7) an expression for the bending moment in each span may be written, from which the maximum positive moments may be determined. Lastly, by formula (4) the strength of the beam may be investigated and by (5) its deflection at any point be deduced.

For example, let the beam in Fig. 36 be regarded as of four equal spans and uniformly loaded with w pounds per linear unit. By a method to be explained in the following articles it may be shown that the bending moments at the supports are,

$$M_2 = -\tfrac{3}{28}wl^2, \qquad M_3 = -\tfrac{2}{28}wl^2, \qquad M_4 = -\tfrac{3}{28}wl^2.$$

From formula (8) the vertical shears at the right of the supports are,

$$V_1 = \tfrac{11}{28}wl, \quad V_2 = \tfrac{15}{28}wl, \quad V_3 = \tfrac{13}{28}wl, \quad \text{etc.}$$

And from (6) those on the left of the supports are found to be,

$$V_1' = -\tfrac{17}{28}wl, \quad V_2' = -\tfrac{13}{28}wl, \quad V_2' = -\tfrac{15}{28}wl, \quad \text{etc.}$$

From formula (7) the general moments now are,

For first span, $M = \tfrac{11}{28}wlx - \tfrac{1}{2}wx^2$,

For second span, $M = -\tfrac{3}{28}wl^2 + \tfrac{15}{28}wlx - \tfrac{1}{2}wx^2$,

For third span, $M = -\tfrac{2}{28}wl^2 + \tfrac{13}{28}wlx - \tfrac{1}{2}wx^2$,

For fourth span, $M = -\tfrac{3}{28}wl^2 + \tfrac{17}{28}wlx - \tfrac{1}{2}wx^2$.

From each of these equations the inflection points may be found by putting $M = 0$, and the point of maximum positive moment by putting $\frac{dM}{dx} = 0$. The maximum positive moments are,

$$\tfrac{121}{1568}wl^2, \quad \tfrac{57}{1568}wl^2, \quad \tfrac{57}{1568}wl^2, \quad \text{and} \quad \tfrac{121}{1568}wl^2.$$

For any particular case the beam may now be investigated by formulas (3) and (4).

The reactions at the supports are not usually needed in the discussion of continuous beams, but if required they may easily be found from the adjacent shears. Thus for the above case,

$$R_1 = 0 + V_1 = \tfrac{11}{28}wl,$$
$$R_2 = V_1^1 + V_2 = \tfrac{32}{28}wl,$$
$$R_3 = V_2^1 + V_3 = \tfrac{26}{28}wl, \quad \text{etc,}$$

and the sum of these is equal to the total load $4wl$.

The equation of the elastic curve in any span is deduced by inserting in (5) for M its value and integrating twice. When $x = 0$, the tangent $\frac{dy}{dx}$ is the tangent of the inclination at the left support and when $x = l$ it is the tangent of the inclination at the right support. When $x = 0$ and also when $x = l$ the ordinate $y = 0$, and from these conditions the two unknown tangents may be found. In general the maximum deflection in any span of a continuous beam will be found intermediate in value between those of a simple beam and a restrained beam.

In the following pages continuous beams will only be investigated for the case of uniform load. The lengths of the spans however may be equal or unequal, and the load per linear foot may vary in the different spans.

Prob. 80. In a continuous beam of three equal spans the negative bending moments at the supports are $\frac{1}{10}wl^2$. Find the inflection points, the maximum positive moments and the reactions of the supports.

ART. 48. THE THEOREM OF THREE MOMENTS.

Let the figure represent any two adjacent spans of a continuous beam whose lengths are l' and l'' and whose uniform loads per linear foot are w' and w'' respectively. Let M', M'' and M''' represent the three unknown moments at the supports. Let V' and V'' be the vertical shears at the right of the first and second supports. Then, for any section distant x from the left support in the first span, the moment is,

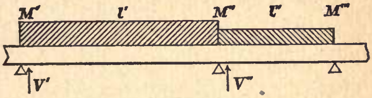


Fig. 37

$$M = M' + V'x - \tfrac{1}{2}wx^2.$$

If this be inserted in the general formula (5) and integrated twice and the constants determined by the condition that $y = 0$ when $x = 0$ and also when $x = l$, the value of the tangent of the angle which the tangent to the elastic curve at any section in the first span makes with the horizontal is found to be,

$$\frac{dy}{dx} = \frac{12M'(2x - l') + 4V'(3x^2 - l'^2) - w'(4x^3 - l'^3)}{24EI}.$$

Similarly if the origin be taken at the next support the value of the tangent of inclination at any point in the second span is,

$$\frac{dy}{dx} = \frac{12M''(2x - l'') + 4V''(3x^2 - l''^2) - w''(4x^3 - l''^3)}{24EI}.$$

Evidently the two curves must have a common tangent at the support. Hence make $x = l'$ in the first of these and $x = 0$ in the second and equate the results, giving,

$$12M'l' + 8V'l'^2 - 3w'l'^3 = -12M''l'' - 4V''l''^2 + w''l''^3.$$

Let the values of V' and V'' be expressed by (8) in terms of M', M'' and M''', and the equation reduces to,

$$(9) \qquad M'l' + 2M''(l' + l'') + M'''l = -\frac{w'l'^3}{4} - \frac{w''l''^3}{4},$$

which is the theorem of three moments for continuous beams uniformly loaded.

If the spans are all equal and the load uniform throughout, this reduces to the simpler form,

$$M' + 4M'' + M''' = -\frac{wl^2}{2}.$$

In any continuous beam of s spans there are $s + 1$ supports and $s - 1$ unknown bending moments at the supports. For each of these supports an equation of the form of (9) may be written containing three unknown moments. Thus there will be stated $s - 1$ equations whose solution will furnish the values of the $s - 1$ unknown quantities.

Prob. 81. A simple beam of oak one inch square and 15 inches long is uniformly loaded with 100 pounds. Find the angle of inclination of the elastic curve at the supports.

Art. 49. Continuous Beams with Equal Spans.

Consider a continuous beam of five equal spans uniformly loaded. Let the supports beginning on the left be numbered 1, 2, 3, 4, 5 and 6. From the theorem of three moments an equation may be written for each of the supports 2, 3, 4, and 5; thus,

$$M_1 + 4M_2 + M_3 = -\tfrac{1}{2}wl^2.$$
$$M_2 + 4M_3 + M_4 = -\tfrac{1}{2}wl^2,$$
$$M_3 + 4M_4 + M_5 = -\tfrac{1}{2}wl^2,$$
$$M_4 + 4M_5 + M_6 = -\tfrac{1}{2}wl^2.$$

Since the ends of the beam rest on abutments without restraint $M_1 = M_6 = 0$. Hence the four equations furnish the means of finding the four moments M_2, M_3, M_4, M_5. The solution may be abridged by the fact that $M_2 = M_5$, and $M_3 = M_4$, which is evident from the symmetry of the beam. Hence,

$$M_2 = M_5 = -\tfrac{4}{38}wl^2, \qquad M_3 = M_4 = -\tfrac{3}{38}wl^2.$$

From formula (8) the shears at the right of the supports are,

$$V_1 = \tfrac{15}{38}\,wl, \qquad V_2 = \tfrac{20}{38}\,wl, \qquad V_3 = \tfrac{19}{38}\,wl, \quad \text{etc.}$$

From (6) the bending moment at any point in any span may now be found as in Art. 47, and by (3), (4) and (5) the complete investigation of any special case may be effected.

In this way the bending moments at the supports for any number of equal spans can be deduced. The following triangular table shows their values for spans as high as seven in number. In each horizontal line the supports are represented by squares in which are placed the coefficients of wl^2. For example, in a beam of 3 spans there are four supports and the bending moments at those supports are 0, $-\frac{1}{10}wl^2$, $-\frac{1}{10}wl^2$, and 0.

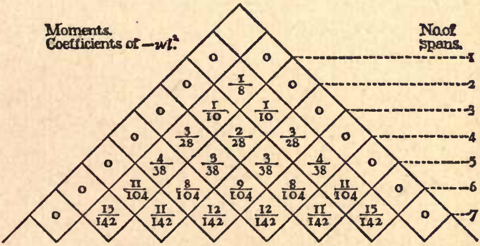


Fig. 38.

The vertical shears at the supports are also shown in the following table for any number of spans up to 5. The space representing a support shows in its left-hand division the shear on the left of that support and in its right-hand division the shear on the right. The sum of the two shears for any support is, of course, the reaction of that support. For example, in a beam of five equal spans the reaction at the second support is $\frac{43}{38}wl$.

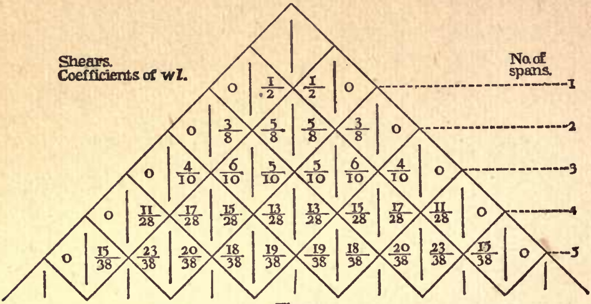


Fig. 39.

It will be seen on examination that the numbers in any oblique column of these tables follow a certain law of increase by which it is possible to extend them, if desired, to a greater number of spans than are here given.

As an example, let it be required to select a **I** beam to span four openings of 8 feet each, the load per span being 500 pounds and the greatest horizontal stress in any fiber to be 12 000 pounds per square inch. The required beam must satisfy formula (4), or,

$$\frac{I}{c} = \frac{M}{12\,000}$$

where M is the maximum moment. From the table it is seen that the greatest negative moment is that at the second support, or $\frac{3}{28}wl^2$. The maximum positive moments are,

For first span, $\quad \max M = \frac{V^2}{2w} = \frac{121}{1568}wl^2,$

For second span, $\max M = M_2 + \frac{V^2}{2w} = \frac{57}{1568}wl^2.$

The greatest value of M is hence at the second support. Then,

$$\frac{I}{c} = \frac{3 \times 500 \times 8 \times 12}{12\,000} = 12$$

and from the table in Art. 30 it is seen that a light 7-inch beam will be required.

Prob. 82. Find what I beam is required to span three openings of 12 feet each, the load on each span being 600 pounds, and the greatest value of S to be 12 000 pounds per square inch.

Prob. 83. Draw the curve of moments and the curve of shears for the case of three equal spans uniformly loaded.

ART. 50. CONTINUOUS BEAMS WITH UNEQUAL SPANS.

As the first example, consider two spans whose lengths are l_1, l_2 and whose loads per linear unit are w_1 and w_2. The theorem of three moments in (9) then reduces to,

$$2M_2(l_1 + l_2) = -\tfrac{1}{4}w_1l_1^3 - \tfrac{1}{4}w_2l_2^3,$$

and hence the bending moment at the middle support is,

$$M_2 = -\frac{w_1l_1^3 + w_2l_2^3}{8(l_1 + l_2)}.$$

From this the reaction at the left support may be found by (8) and the bending moment at any point by (6).

Next consider three spans whose lengths are l_1, l_2 and l_3, loaded uniformly with w_1, w_2, w_3. The bending moments at the second and third supports are M_2 and M_3. Then from (9),

$$2M_2(l_1 + l_2) + M_3l_3 = -\tfrac{1}{4}w_1l_1^3 - \tfrac{1}{4}w_2l_2^3,$$
$$M_2l_2 + 2M_3(l_2 + l_3) = -\tfrac{1}{4}w_2l_2^3 - \tfrac{1}{4}w_3l_3^3,$$

and the solution of these gives the values of M_2 and M_3. A very common case is that for which $l_2 = l$, $l_1 = l_3 = nl$, and $w_1 = w_2 = w_3 = w$. For this case the solution gives,

$$M_2 = M_3 = -\frac{1 + n^3}{2 + 3n} \cdot \frac{wl^2}{4}.$$

Here if $n = 1$ the moments become $-\frac{wl^3}{10}$ as shown in the last article.

Whatever be the lengths of the spans or the intensity of the loads, the theorem of three moments furnishes the means of finding the bending moments at the supports. Then from (8), (7) and (6) the vertical shears and bending moments at every section may be computed. Finally, if the material be not strained beyond its elastic limit, formula (5) may be used to determine the stiffness, while (4) investigates the strength of the beam.

Prob. 84. A heavy 12-inch I beam of 36 feet length covers four openings, the two end ones being each 8 feet and the others each 10 feet in span. Find the maximum moment in the beam. Then determine the load per linear foot so that the greatest horizontal unit-stress may be 12 000 pounds per square inch.

Prob. 85. A continuous beam of three equal spans is loaded only in the middle span. Prove that the reactions of the end supports due to this load are $-\frac{wl}{20}$.

ART. 51. REMARKS ON THE THEORY OF FLEXURE.

The theory of flexure presented in this and the preceding chapter is called the common theory, and is the one universally adopted for the practical investigation of beams. It should not be forgotten, however, that the axioms and laws upon which it is founded are only approximate and not of an exact nature like those of mathematics. Laws (A) and (B) for instance are true as approximate laws of experiment, but not as exact laws of science. Law (G) indeed rests upon so slight experimental evidence that it is more of a hypothesis than an established truth. Objections may also be raised against the validity of the method of resolving the internal stresses into their horizontal and vertical components, and against the formula (3) which supposes the vertical shear to be uniformly distributed over the cross-section.

When experiments on beams are carried to the point of rupture and the longitudinal unit-stress S computed from formula (4) a disagreement of that value with those found by direct experiments on tension or compression is observed. This is often regarded as an objection to the common theory of flexure, but it is in reality no objection, since law (G) and formula (4) are only true provided the elastic limit of the material be not exceeded. Experiments on the deflection of beams furnish on the other hand the most satisfactory confirmation of the theory. When E is known by tensile or compressive tests the formulas for deflection are found to give values closely agreeing with those observed. Indeed so reliable are these formulas that it is not uncommon to use them for the purpose of computing E from experiments on beams. If however the elastic limit of the material be exceeded, the computed and observed deflections fail to agree.

On the whole it may be concluded that the common theory of flexure is entirely satisfactory and sufficient for the investigation of all practical questions relating to the strength and stiffness of beams. The actual distribution of the internal stresses is however a matter of very much interest and this will be discussed in Chapter VIII.

The theory of flexure is here applied to continuous beams only for the case of uniform loads. It should be said however that there is no difficulty in extending it to the case of concentrated loads. By a course of reasoning similar to that of Art. 48 it may be shown that the theorem of three moments for single loads is,

$$M'l' + 2M''(l' + l'') + M'''l'' = -P'l'^2(k - k^3)$$
$$- P''l''^2(2k - 3k^2 + k^3.)$$

Here as in Fig. 37 the moments at three consecutive supports are designated by M', M'' and M''' and the lengths of the two spans by l' and l''. P' is any load on the first span at a distance kl' from the left support and P'' any load on the second span at a distance kl'' from the left support, k being any fraction less than unity and not necessarily the same in the two cases. From this theorem the negative bending moments at the supports for any concentrated loads may be found, and the beam be then investigated by formulas (6) and (4). For example, if a beam of three equal spans be loaded with P at the middle of each span the negative moments at the supports are each $\frac{3}{20}Pl$.

The Journal of the Franklin Institute for March and April, 1875, contains an article by the author in which the law of increase of the quantities in the tables of Art. 45 is explained and demonstrated. A general abbreviated method of deducing the moments at the supports for both uniform and concentrated loads on restrained and continuous beams is given in the Philosophical Magazine for September, 1875. See also Van Nostrand's Science Series, No. 25.

Exercise 4. Procure six sticks of ash each ⅛ × ⅜ inches and of lengths 1, 2 and 3 feet. Devise and conduct experiments to test the following laws: First, the strength of a beam varies directly as its breadth and directly as the square of its depth. Second, the stiffness of a beam is directly as its breadth and directly as the cube of its depth. Third, a beam fixed at the ends is twice as strong and four times as stiff as a simple beam when loaded at the middle. Write a report describing and discussing the experiments.

Exercise 5. Consult Barlow's Strength of Materials (London, 1837,) and write an essay concerning his experiments to determine the laws of the strength and stiffness of beams. Consult also Ball's Experimental Mechanics.

Exercise 6. In order to test the theory of continuous beams discuss the following experiments by Francis and ascertain whether or not the ratio of the two observed deflections agrees with theory. "A frame was erected, giving 4 bearings in the same horizontal plane, 4 feet apart, making 3 equal spans, each bearing being furnished with a knife edge on which the beam was supported. Immediately over the bearings and secured to the same frame was fixed a straight edge, from which the deflections were measured. A bar of common English refined iron, 12 feet 2¾ inches long, mean width 1.535 inches, mean depth 0.367

inches, was laid on the 4 bearings, and loaded at the center of each span so as to make the deflections the same, the weight at the middle span being 82.84 pounds and at each of the end spans 52.00 pounds. The deflections with these weights were,

At the center of the middle span 0.281 inches.
At center of end span, 0.275 and 0.284 inches,
mean 0.280 inches.

A piece 3 feet 11½ inches long was then cut from each end of the bar, leaving a bar 4 feet 4¾ inches long, which was replaced in its former position and loaded with the same weight (82.84 pounds) as before, when its deflection was found to be 1.059 inches."

Prob. 86. A beam of three spans, the center one being l and the side ones nl, is loaded with P at the middle of each span. Find the value of n so that the reactions may be equal.

CHAPTER V.

ON THE COMPRESSION OF COLUMNS.

Art. 52. Cross-Sections of Columns.

A column is a prism, greater in length than about ten times its least diameter, which is subject to compression. If the prism be only about four or six times as long as its least diameter the case is one of simple compression, the constants for which are given in Art. 6. In a case of simple compression failure occurs by the crushing and splintering of the material, or by shearing in directions oblique to the length. In the case of a column, however, failure is apt to occur by a sidewise bending which induces transverse stresses and causes the material to be highly strained under the combined compression and flexure.

Wooden columns are usually square or round and they may be built hollow. Cast iron columns are usually round and they are often cast hollow. Wrought iron columns are made of a great variety of forms. A I may be used as a column, but they are usually made of three or more different shape-irons riveted together. The Phœnix column is made by riveting together flanged circular segments so as to form a closed cylinder. It is evident that a square or round section is preferable to an unsymmetrical one, since then the liability to bending is the same in all directions. For a rectangular section the plane of flexure will evidently be perpendicular to the longer side of the cross-section, and in general the plane of flexure will be perpendicular to that axis of the cross-section for which the moment of inertia is the least. In designing a column it is hence advisable that the cross-section should be so arranged that the moments of inertia about the two principal rectangular axes may be equal.

For instance, let it be required to construct a column with two I shapes and two plates as shown in Fig. 40. The I beams are to be light 10-inch ones weighing 30 pounds per linear foot, and having the flanges 4.32 inches wide. The plates are to be ½ inch thick and it is required to find their length x so that the liability to bending about the two axes shown in the figure may be the same. From the table in Fig. 30 it is ascertained that the moment of inertia I of the beam about an axis through its center of gravity and perpendicular to the web is 150, while the moment of inertia I' about an axis through the same point and parallel to the web is nearly 8. Hence, for the axes shown in Fig. 40, the moments of inertia are,

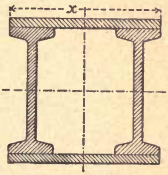


Fig. 40.

For axis perpendicular to plates,

$$2\frac{0.5 \times x^3}{12} + 2 \times 8 + 2 \times 9 \times \left(\frac{x}{2} - 2.16\right)^2.$$

For axis parallel to plates,

$$2\frac{x \times 0.5^3}{12} + 2 \times 0.5x \times 5.25^2 + 2 \times 150.$$

Placing these two expressions equal, the value of x is found to be between 10½ and 11 inches.

Prob. 87. A column is to be formed of two light 12-inch eye-beams connected by a lattice bracing. Find the proper distance between their centers, disregarding the moment of inertia of the latticing.

Prob. 88. Two joists each 2 × 4 inches are to be placed 6 inches apart between their centers, and connected by two others each 8 inches wide and x inches thick so as to form a closed hollow rectangular column. Find the proper value of x.

Art. 53. General Principles.

If a short prism of cross-section A be loaded with the weight P, the internal stress is to be regarded as uniformly distributed over the cross-section, and hence the compressive unit-stress S_c is $\frac{P}{A}$. But for a long prism, or column, this is not the case; while the average unit-stress is $\frac{P}{A}$, the stress in certain parts of the cross-section may be greater and upon others less than this value on account of the transverse stresses due to the sidewise flexure. Hence in designing a column the load P must be taken as smaller for a long one than for a short one, since evidently the liability to bending increases with the length.

Numerous experiments on the rupture of columns have shown that the load causing the rupture is approximately inversely proportional to the length of the column. That is to say, if there be two columns of the same material and cross-section and one twice as long as the other, the long one will rupture under about one-quarter the load of the short one.

The condition of the ends of columns exerts a great influence upon their strength. Class (a) includes those with 'round ends,' or those in such condition that they are free to turn at the ends. Class (c) includes those whose ends are 'fixed' or in such condition that the tangent to the curve at the ends always remains vertical. Class (b) includes those with one end fixed and the other round. In architecture it is rare that any other than class (c) is used. In bridge construction and in machines, however, columns of classes (b) and (a) are very common.

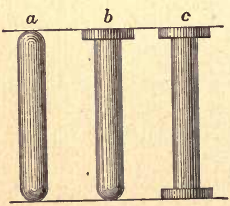


Fig. 41.

It is evident that class (c) is stronger than (b), and that (b) is stronger than (*c*), and this is confirmed by all experiments. Fig. 41 is intended as a symbolical representation of the three classes of columns, and not as showing how the ends are rendered 'round' and 'fixed' in practical constructions.

The theory of the resistance of columns has not yet been perfected like that of beams, and accordingly the formulas for practical use are largely of an empirical character. The form of the formulas however is generally determined from certain theoretical considerations, and these will be presented in the following articles as a basis for deducing the practical rules.

Prob. 89. A column formed of two **I** beams each weighing 93 pounds per yard is 11 inches square and 3 feet long. What load will it carry with a factor of safety of 5?

ART. 54. EULER'S FORMULA.

Consider a column of cross-section A loaded with a weight P under whose action a certain small sidewise bending occurs. Let the column be round or free to turn at both ends as in Fig. 42. Take the origin at the upper end, and let x be the vertical and y the horizontal co-ordinate of any point of the elastic curve. The general equation (4), deduced in Art. 33, applies to all bodies subject to flexure provided the bending be slight and the elastic limit of the material be not exceeded. For the column the bending moment is $-Py$, and hence,

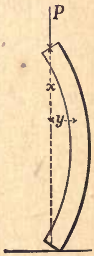


Fig. 42.

$$EI\frac{d^2y}{dx^2} = -Py.$$

The first integration of this gives,

$$EI\frac{dy^2}{dx^2} = -Py^2 + C.$$

But when $y =$ the maximum deflection Δ, the tangent $\frac{dy}{dx} = 0$.

Hence $C = P\Delta^2$, and by inversion,

$$dx = \left(\frac{EI}{P}\right)^{\frac{1}{2}} \frac{dy}{\sqrt{\Delta^2 - y^2}}.$$

The second integration now gives,

$$x = \left(\frac{EI}{P}\right)^{\frac{1}{2}} \text{arc sin} \frac{y}{\Delta} + C'.$$

Here C' is 0 because $y = 0$ when $x = 0$. Hence finally the equation of the elastic curve of the column is,

$$y = \Delta \sin x\left(\frac{P}{EI}\right)^{\frac{1}{2}}.$$

This equation is that of a sinusoid. But also $y = 0$ when $x = l$. Hence if n be an integer, $l\left(\frac{P}{EI}\right)^{1/2}$ must equal $n\pi$, or,

$$P = EI\frac{n^2\pi^2}{l^2},$$

which is Euler's formula for the resistance of columns. This reduces the equation of the sinusoid to,

$$y = \Delta \sin n\pi\frac{x}{l}.$$

The three curves for $n = 1$, $n = 2$, and $n = 3$ are shown in Fig. 43. In the first case the curve is entirely on one side of the axis of x, in the second case it crosses that axis at the middle, and in the third case it crosses at $\frac{1}{3}l$ and $\frac{2}{3}l$, the points of crossing being also inflection points where the bending moment is zero. Evidently the greatest deflection will occur for the case where $n = 1$, and this is the most dangerous case. Hence,

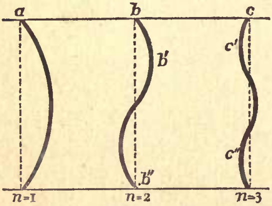


Fig. 43.

$$(a) \qquad P = \frac{\pi^2 EI}{l^2},$$

is Euler's formula for columns with round ends.

A column with one end fixed and the other round is closely represented by the portion $b'b''$ of the second case, b' being the fixed end where the tangent to the curve is vertical. Here $n = 2$, and the length $b'b''$ is three-fourths of the entire length, hence,

$$(b) \qquad P = \tfrac{9}{4}\frac{\pi^2 EI}{l^2}$$

is Euler's formula for columns with one end fixed and the other round.

A column with fixed ends is represented by the portion $c'c''$ of the case c. Here $n = 3$, and the length $c'c''$ is three-fourths of the entire length, hence,

$$(c) \qquad P = 4\frac{\pi^2 EI}{l^2}$$

is Euler's formula for columns with fixed ends.

From this investigation it appears that the relative resistances of three columns of the classes (a), (b) and (c) are as the numbers 1, 2¼ and 4, when the lengths are the same, and this conclusion is approximately verified by experiments. It also appears that, if the resistance of three columns of the classes (a), (b) and (c) are to be equal, their lengths must be as the numbers 1, 1½ and 2.

The moment of inertia I in the above formulas is taken about a neutral axis of the cross-section perpendicular to the plane of the flexure, and in general is the least moment of inertia of that cross-section, since the column will bend in the direction which offers the least resistance. For a rectangular column whose greatest side is b and least side d, the formulas may be written,

$$P = \frac{m\pi^2 Ebd^3}{12l^2}, \qquad \text{where } m = 1,\ 2\tfrac{1}{4},\ \text{or } 4.$$

For a cylindrical column of diameter d the formulas are,

$$P = \frac{m\pi^3 Ed^4}{64l^2}, \qquad \text{where } m = 1,\ 2\tfrac{1}{4},\ \text{or } 4.$$

Hence the strength of a column varies directly as its cross-section and directly as the square of its least diameter or side. In general if r be the least radius of gyration of the cross-section the value of I is Ar^2 and the formula may be written,

$$\frac{P}{A} = \frac{m\pi^2 Er^2}{l^2}, \qquad \text{where } m = 1, 2\tfrac{1}{4}, \text{ or } 4.$$

which shows that P varies as the square of the ratio of r to l.

The maximum deflection Δ is indeterminate, so that the load P, given by Euler's formula, is merely the load which causes the column to bend. Practically the bending of a column is the beginning of its failure.

Prob. 90. Show that Euler's formula for the case of a column fixed at one end and entirely free at the other is $\dfrac{P}{A} = \dfrac{\pi^2 Er^2}{4l^2}$.

Art. 55. Hodgkinson's Formulas.

Euler's formula gives valuable information regarding the laws of flexure of columns, but is difficult of direct practical application because it indicates no relation between the load P and the greatest internal compressive unit-stress. It shows that the strength of cylindrical columns varies directly as the fourth power of the diameter and inversely as the square of the length. Hodgkinson in his experiments observed that this was approximately but not exactly the case. He therefore wrote for each kind of columns the analogous expression,

$$P = \alpha \frac{d^\beta}{l^\delta}$$

and determined the constants α, β and δ from the results of his experiments, thus producing empirical formulas.

Let P be the crushing load in gross tons, d the diameter of the column in inches, and l its length in feet. Then Hodgkinson's empirical formulas are,

For solid cast iron cylindrical columns,

$$P = 14.9\frac{d^{3.5}}{l^{1.63}} \qquad \text{for round ends,}$$

$$P = 44.2\frac{d^{3.5}}{l^{1.63}} \qquad \text{for flat ends,}$$

For solid wrought iron cylindrical columns,

$$P = 42\frac{d^{3\cdot76}}{l^2} \qquad \text{for round ends,}$$

$$P = 134\frac{d^{3\cdot76}}{l^2} \qquad \text{for flat ends.}$$

These formulas indicate that the ultimate strength of flat-ended columns is about three times that of round-ended ones. The experiments also showed that the strength of a column with one end flat and the other end round is about twice that of one having both ends round. Hodgkinson's tests were made upon small columns and his formulas are not so reliable as those which will be given in the following articles. For small cast iron columns however the formulas are still valuable.

By the help of logarithms it is easy to apply these formulas to the discussion of given cases. Usually P will be given and d required, or d be given and P required. By using assumed factors of safety the proper size of cylindrical columns to carry given loads may also be determined. These formulas, it should be remembered, do not apply to columns shorter than about thirty times their least diameters. The word flat used in this article is to be regarded as equivalent to fixed.

Prob. 91. A cast iron cylindrical column with flat ends is 3 inches diameter and 8 feet long. What load will cause it to fail?

Prob. 92. A cast iron cylindrical column with flat ends is to be 7 feet long and carry a load of 200 000 pounds with a factor of safety of 6. Find the proper diameter.

ART. 56. TREDGOLD'S FORMULA.

The formulas of Euler are defective because they contain no constant indicating the working or ultimate compressive strength of the material and because they apply only to long columns. Hodgkinson's formulas are unsatisfactory for similar reasons and because they do not well represent the results of later experiments, Tredgold's formula was deduced to remedy these defects. It may be established by the following considerations.

Let the column be of rectangular section, the area being A, the least side d, the greatest side b, and the length l. Let P be the load upon it. The average compressive unit-stress at any section is $\frac{P}{A}$, but in consequence of the sidewise deflection this is increased on the concave side and decreased on the convex side by an amount S. From the fundamental equation (4) the value of S is $\frac{6M}{bd^2}$, and if Δ be the maximum deflection the greatest value of S is $\frac{6P\Delta}{bd^2}$. Now if S_c be the total compressive unit-stress on the concave side $S = S_c - \frac{P}{A}$, and hence,

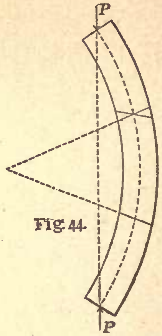


Fig. 44.

$$S_c - \frac{P}{A} = \frac{6P\Delta}{bd^2} = \frac{6P\Delta}{Ad}.$$

Accordingly the value of S_c is

$$S_c = \frac{P}{A} + \frac{P}{A} \cdot \frac{6\Delta}{d}.$$

The value of Δ is unknown, but if the curve of deflection be an arc of a circle, which it is very nearly, Δ equals approximately $\frac{l^2}{8R}$, in which R represents the radius of curvature of the column. Now, as in Art. 33, the value of R for the same unit-stress S varies directly as d and inversely as E. Hence Δ may be taken as varying directly as l^2 and inversely as d. Accordingly if k be a number depending upon the kind of material and the arrangement of the ends of the column, the value of S_c may be written,

$$S_c = \frac{P}{A} + k \frac{P}{A} \frac{l^2}{d^2}.$$

From this the value of the unit load $\frac{P}{A}$ is,

$$\frac{P}{A} = \frac{S_c}{1 + k\frac{l^2}{d^2}},$$

which is Tredgold's formula for resistance of columns.

The quantity k can not be determined theoretically. As the above reasoning shows, its value varies with the form of cross-section as well as with the kind of material and the arrangement of the ends of the column. For instance, the value of k is not the same for a circular section with diameter d as for a rectangular section whose least side is d. It is however not uncommon to find this formula stated as applicable to any cross-section whose least diameter is d.

In order to determine k recourse must be had to experiments. These are usually conducted by loading columns to the point of rupture. P, A, l and d are known and thus the constants S_c and k may be computed. Theoretically S_c is the ultimate compressive strength of the material and the values found for it by experiments on columns agree roughly with those deduced by the direct crushing of short specimens. The value of k is always less than unity and it is subject to great variation, even in columns of the same material. For a column with round ends k was regarded by Tredgold and Gordon as being four times as great as for a column with fixed ends, since both experiment and theory indicate that a fixed-ended column of length $2l$ has the same strength as a round-ended column of length l. Therefore for the ultimate strength of columns,

For fixed ends, $$\frac{P}{A} = \frac{S_c}{1 + k\frac{l^2}{d^2}},$$

For round ends, $$\frac{P}{A} = \frac{S_c}{1 + 4k\frac{l^2}{d^2}}.$$

The following values of S_c and k were deduced by Gordon from Hodgkinson's experiments, and are given by Rankine,

For stone and brick, $S_c =$ see Art. 6, $k = \frac{1}{600}$,

For timber (rectangular sections), $S_c = 7\,200$, $k = \frac{1}{250}$,

For cast iron cylinders, $S_c = 80\,000$, $k = \frac{1}{800}$,

For wrought iron (rectangular sections), $S_c = 36\,000$, $k = \frac{1}{3000}$.

These values of S_c are in pounds per square inch, while those of k are abstract numbers.

Tredgold's formula is sometimes used under the name of Gordon's formula, The reasoning by which it is deduced is not entirely satisfactory and it often fails to properly represent the results of experiment.

Prob. 93. Find the values of S_c and k from the two following experiments on flat-ended Phœnix columns. The sectional area of each column was 12 square inches and the exterior diameter 8 inches. The length of the first column was 25 feet and it failed under a load of 420 000 pounds. The length of the second column was 10 feet and it failed under a load of 478 000 pounds.

Art. 57. Gordon's Formula.

The formula which seems to most satisfactorily represent the results of experiments will now be deduced. It is often called Gordon's formula and sometimes Rankine's formula, and occasionally it is referred to as "Gordon's formula modified by Rankine." It does not appear however that either Gordon or Rankine developed it in its general form or used it for the discussion of experiments. The name of Gordon will here be applied to it because that is most frequently used and because of the lack of a better appellation. It is similar to Tredgold's formula, but has the advantage of being applicable to any form of cross-section.

Let P be the load on the column, l its length, A the area of its cross-section, I the moment of inertia and r the radius of gyration of that cross-section with reference to a neutral axis perpendicular to the plane of flexure, and c the shortest distance from that axis to the remotest fiber on the concave side. The average compressive unit-stress on any cross-section is $\frac{P}{A}$ but in consequence of the flexure this is increased on the concave side and decreased on the convex side. Thus in Fig. 45 the average unit-stress $\frac{P}{A}$ is represented by ab, but on the concave side this is increased to cg and on the convex side decreased to eh. The triangles bdg and bfh represent the effect of the flexure exactly as in the case of beams, dg indicating the greatest compressive and hf the greatest tensile unit-stress due to the bending. Let the total maximum unit-stress be denoted by S_c and the part due to the flexure be denoted by S. Then,

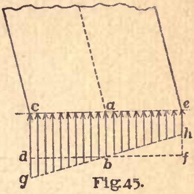


Fig. 45.

$$S_c = \frac{P}{A} + S.$$

Now, from the fundamental formula (4), the flexural stress is $\frac{Mc}{I}$, where M is the external bending moment, which for a column has its greatest value when $M = P\Delta$, Δ being the maximum deflection. $I = Ar^2$ is the well known relation between I and r. Hence the value of S is,

$$S = \frac{P\Delta c}{I} = \frac{P\Delta c}{Ar^2}.$$

By analogy with the theory of beams, as in Art. 37, the value of Δ may be regarded as varying directly as $\frac{l^2}{c}$. Hence if q be a quantity depending upon the kind of material and the condition of the ends, the total unit-stress is,

$$S_c = \frac{P}{A} + q\frac{Pl^2}{Ar^2}.$$

This may now be written in the usual form,

$$(10) \qquad \frac{P}{A} = \frac{S_c}{1 + q\frac{l^2}{r^2}}.$$

which is Gordon's formula for the investigation of columns.

The above reasoning has been without reference to the arrangement of the ends of the column. By Art. 54 it is known that a column with round ends must be one-half the length of one with fixed ends in order to be of equal strength, and that a column with one end fixed and the other round must be three-fourths the length of one with fixed ends in order to be of equal strength. Therefore if q be the constant for fixed ends, $(\frac{4}{3})^2 q$ will be the constant for one end fixed and the other round, and $2^2 q$ will be the constant for both ends round.

The values of q to be taken for use in formula (10) for the examples and problems of this chapter may be the following rough values, unless otherwise stated, while the values of the ultimate compressive unit-stress S_c will be taken from Art. 6.

Material.	Both Ends Fixed.	Fixed and Round.	Both Ends Round.
Timber	$\frac{1}{3\ 000}$	$\frac{1.78}{3\ 000}$	$\frac{4}{3\ 000}$
Cast Iron	$\frac{1}{5\ 000}$	$\frac{1.78}{5\ 000}$	$\frac{4}{5\ 000}$
Wrought Iron	$\frac{1}{36\ 000}$	$\frac{1.78}{36\ 000}$	$\frac{4}{36\ 000}$
Steel	$\frac{1}{25\ 000}$	$\frac{1.78}{25\ 000}$	$\frac{4}{25\ 000}$

The very wide variation in the values of q found from different experiments shows however that little dependence can be placed upon average results. In any practical case of importance an effort should be made to ascertain values of S_c and q for the special kind of columns on hand.

Prob. 94. Plot the curve represented by formula (10) for cases of wrought iron columns with fixed and with round ends, taking the values of $\frac{P}{A}$ as ordinates and the values of $\frac{l}{r}$ as abscissas.

ART. 58. RADIUS OF GYRATION OF CROSS-SECTIONS.

The radius of gyration of a surface with reference to an axis is equal to the square root of the ratio of the moment of inertia of the surface referred to the same axis to the area of the figure. Or if r be radius of gyration, I the moment of inertia and A the area of the surface, then $I = Ar^2$.

In the investigation of columns by formula (10) the value of r^2 is required, r being the least radius of gyration. These values are readily derived from the expressions for the moment of inertia given in Art. 22, the most common cases being the following.

For a rectangle whose least side is d, $\quad r^2 = \frac{d^2}{12}$.

For a circle of diameter d, $\quad r^2 = \frac{d^2}{16}$.

For a triangle whose least altitude is d, $\quad r^2 = \frac{d^2}{18}$.

For a hollow square section, $\quad r^2 = \frac{d^2 + d'^2}{12}$.

For a hollow circular section, $\quad r^2 = \frac{d^2 + d'^2}{16}$.

For I beams and other shapes r^2 is found by dividing the least moment of inertia of the cross-section by the area of that cross-section. For instance, by the help of the table in Art. 30, the least value of r^2 for a light 12-inch eye beam is found to be $\frac{11.0}{12.6} = 0.87$ inches^2.

Prob. 95. An angle iron is $3 \times 4 \times 0.5$ inches. Find the least moment of inertia and least radius of gyration.

ART. 59. INVESTIGATION OF COLUMNS.

The investigation of a column consists in determining the maximum compressive unit-stress S_c from formula (10). The values of P, A, l and r will be known from the data of the given case and q is known from the results of previous experiments. Then,

$$S_c = \frac{P}{A}\left(1 + q\frac{l^2}{r^2}\right),$$

and, by comparing the computed value of S_c with the ultimate strength and elastic limit of the material, the factor of safety and the degree of stability of the column may be inferred.

For example, consider a hollow cast iron column of rectangular section, the outside dimensions being 4×5 inches and the inside dimensions 3×4 inches. Let the length be 18 feet, the ends fixed, and the load be 80 000 pounds. Here $P = 80\,000$, $A = 8$ square inches, $l = 216$ inches. From the table $q = \frac{1}{3\,000}$. From Art. 58,

$$r^2 = \frac{5 \times 4^3 - 4 \times 3^3}{12 \times 8} = 2.2.$$

Then the substitution of these values gives,

$$S_c = \frac{80\,000}{8}\left(1 + \frac{216 \times 216}{5\,000 \times 2.2}\right) = 42\,000 \text{ pounds per square in.}$$

Here the average unit-stress is 10 000 pounds per square inch, but the flexure has increased that stress on the concave side to 42 000 pounds per square inch so that the factor of safety is only about two.

Prob. 96. A cylindrical wrought iron column with fixed ends is 12 feet long, 6.36 inches in exterior diameter, 6.02 inches in interior diameter, and carries a load of 98 000 pounds. Find its factor of safety.

Prob. 97. A pine stick 3×3 inches and 12 feet long is used as a column with fixed ends. Find its factor of safety under a load of 3 000 pounds. If the length be only one foot, what is the factor of safety.

ART. 60. SAFE LOADS FOR COLUMNS.

To determine the safe load for a given column it is necessary to first assume the allowable working unit-stress S_c. Then from formula (10) the safe load is,

$$P = \frac{S_c A}{1 + q\frac{l^2}{r^2}}.$$

Here A, l and r are known from the data of the given problem and q is taken from the table in Art. 58.

For example, let it be required to determine the safe load for a fixed-ended timber column, 3 × 3 inches square and 12 feet long, so that the greatest compressive unit-stress may be 800 pounds per square inch. From the formula,

$$P = \frac{800 \times 9}{1 + \frac{12^5}{3\,000 \times 3^2}} = \text{about } 700 \text{ pounds.}$$

A short prism 3 × 3 inches should safely carry ten times this load.

Prob. 98. Find the safe load for a heavy wrought iron **I** of 15 inches depth and for lengths of 5, 10 and 15 feet when used as columns with fixed ends.

ART. 61. DESIGNING OF COLUMNS.

When a column is to be selected or designed the load to be borne will be known, as also its length and the condition of the ends. A proper allowable unit-stress S_c is assumed, suitable for the given material under the conditions in which it is used. Then from formula (1) the cross-section of a short column or prism is $\frac{P}{S_c}$ and it is certain that a greater value of the cross-section than this will be required. Next assume a form and area A, find r^2, and from the formula (10) compute S_c. If the computed value agrees with the assumed value the correct size has been selected.

If not, assume a new area and compute S_c again, and continue the process until a proper agreement is attained.

For example, a hollow cast iron rectangular column of 18 feet length is to carry a load of 60 000 pounds. Let the working strength S_c be 15 000 pounds per square inch. Then for a short length the area required would be four square inches. Assume then that about 6 square inches will be needed. Let the section be square, the exterior dimensions 6 × 6 inches, and the interior dimensions $5\frac{1}{2} \times 5\frac{1}{2}$ inches. Then $A = 5.75$, $l = 18 \times 12$, $P = 60\,000$, $q = \frac{1}{5\,000}$, $r^2 = 5.52$, and from (10),

$$S_c = \frac{60\,000}{5.75}\left(1 + \frac{18^2 \times 12^2}{5\,000 \times 5.52}\right) = \text{about } 30\,000,$$

which shows that the dimensions are much too small.

Again assume the exterior side as 6 inches and the interior as 5 inches. Then $A = 11$, $r^2 = 5.08$, and

$$S_c = \frac{60\,000}{11}\left(1 + \frac{18^2 \times 12^2}{5\,000 \times 5.08}\right) = \text{about } 15\,700.$$

As this is very near the required working stress it appears that these dimensions very nearly satisfy the imposed conditions.

In many instances it is possible to assume all the dimensions of the column except one and then after expressing A and r in terms of this unknown quantity to introduce them into (10) and solve the problem by finding the root of the equation thus formed. For example let it be required to find the size of a square wooden column with fixed ends and 24 feet long to sustain a load of 100 000 pounds with a factor of safety of 10. Here let x be the unknown side; then $A = x^2$ and $r^2 = \frac{x^2}{12}$. From (10),

$$800 = \frac{100\,000}{x^2}\left(1 + \frac{24^2 \times 12^3}{3\,000 \times x^2}\right).$$

By reduction this becomes,

$$8x^4 - 1\,000x^2 = 251\,776,$$

the solution of which gives 14.6 inches for the side of the column.

Prob. 99. A hollow cylindrical cast iron column is to be designed to carry a load of 200 000 pounds. Its length is to be 12 feet, its ends flat or fixed, its exterior diameter 6 inches and the allowable unit-stress 15 000 pounds per square inch. Find the proper interior diameter.

Prob. 100. Find the size of a square wooden column with fixed ends and 12 feet in length to sustain a load of 100 000 pounds with a factor of safety of 10. Find also its size for round ends.

ART. 62. EXPERIMENTS ON COLUMNS.

It is impossible to present here even a summary of the many experiments that have been made to determine the laws of resistance of columns. The interesting tests made by Christie in 1883 for the Pencoyd Iron Works will however be briefly described on account of their great value and completeness as regards wrought iron struts, embracing angle, tee, beam and channel sections. See Transactions of the American Society of Civil Engineers, April, 1884.

The ends of the struts were arranged in different methods; first flat ends between parallel plates to which the specimen was in no way connected; second, fixed ends, or ends rigidly clamped; third, hinged ends, or ends fitted to hemispherical balls and sockets or cylindrical pins; fourth, round ends, or ends fitted to balls resting on flat plates.

The number of experiments was about 300, of which about one-third were upon angles, and one-third upon tees. The quality of the wrought iron was about as follows: elastic limit 32 000 pounds per square inch, ultimate tensile strength 49 600 pounds per square inch, ultimate elongation 18 per cent in 8 inches. The length of the specimens varied from 6 inches up to 16 feet, and the ratio of length to least radius of gyration varied from 20 to 480. Each specimen was placed in a Fairbanks' testing machine of 50 000 pounds capacity and the power applied by hand through a system of gearing to two rigidly parallel plates between which the specimen was placed in a vertical position. The pressure or load was measured on an ordinary scale beam, pivoted on knife edges and carrying a moving weight which registered the pressure automatically. At each increment of 5 000 pounds, the lateral deflection of the column was measured. The load was increased until failure occured.

The following are the combined average results of these carefully conducted experiments. The first column gives the values of $\frac{l}{r}$, and the other columns the value of $\frac{P}{A}$ or the ultimate load per inch of cross-section. From these results it will be seen

Length divided by Least Rad. of Gyration.	Flat Ends.	Fixed Ends.	Hinged Ends.	Round Ends.
20	46 000	46 000	46 000	44 000
40	40 000	40 000	40 000	36 500
60	36 000	36 000	36 000	30 500
80	32 000	32 000	31 500	25 000
100	29 800	30 000	28 000	20 500
120	26 300	28 000	24 300	16 500
140	23 500	25 500	21 000	12 800
160	20 000	23 000	16 500	9 500
180	16 800	20 000	12 800	7 500
200	14 500	17 500	10 800	6 000
220	12 700	15 000	8 800	5 000
240	11 200	13 000	7 500	4 300
260	9 800	11 000	6 500	3 800
280	8 500	10 000	5 700	3 200
300	7 200	9 000	5 000	2 800
320	6 000	8 000	4 500	2 500
340	5 100	7 000	4 000	2 100
360	4 300	6 500	3 500	1 900
380	3 500	5 800	3 000	1 700
400	3 000	5 200	2 500	1 500
420	2 500	4 800	2 300	1 300
440	2 200	4 300	2 100	
460	2 000	3 800	1 900	
480	1 900		1 800	

that when the strut is short there is no practical difference in the strength of the four classes, and that when the strut is long there is but little difference between those with flat and hinged ends. The strength of the long columns with fixed ends appears to be about 3½ times that of the round-ended ones.

Prob. 101. Plot the above experiments, taking the values of $\frac{l}{r}$ as abscissas and those of $\frac{P}{A}$ as ordinates.

Prob. 102. What load will cause the rupture of a wrought iron strut of an angle section 1 × 1 × ⅛ inch and 5 feet long when acting with flat ends? Ans. About 7 200 pounds.

ART. 63. ON THE THEORY OF COLUMNS.

It has been already remarked that the theory of columns is in a very incomplete condition compared with that of beams. A satisfactory formula for the resistance of columns should be of such a nature that for a short block which fails by pure crushing it would reduce to the equation $P = AS_c$, while for a long strut which fails by bending it would reduce to an expression like Euler's. The formula of Gordon conforms partly to this requirement, but the fact that it is impossible to determine values of q of general applicability indicates that q is not a constant, and that the reasoning by which it is deduced is faulty. Nevertheless Gordon's formula applies so well to columns of medium length that it is extensively employed in this country in the manner illustrated in the preceding articles.

For long columns Euler's formula often represents fairly the results of experiments, and since it contains I it may be adapted to any form of cross-section. Thus $I = Ar^2$, and,

For round ends, $$\frac{P}{A} = \frac{\pi^2 E r^2}{l^2},$$

For fixed ends, $$\frac{P}{A} = \frac{4\pi^2 E r^2}{l^2}.$$

For wrought iron E equals about 25 000 000 pounds per square inch and hence for round ends,

$$\text{if} \quad \frac{l}{r} = \quad 200, \qquad 300, \qquad 400,$$

$$\frac{P}{A} = 6\,250, \qquad 2\,800, \qquad 1\,600,$$

and these agree well with the experimental values given in the last article.

In conclusion it may be well to show that Gordon's formula when properly deduced is essentially of the same form as Euler's, and that the number q cannot be a true constant. For this purpose consider the reasoning of Art. 57, and as there take the total compressive unit-stress S_c on the concave side as equal to the sum of the average unit-stress and the flexural unit-stress, or,

$$S_c = \frac{P}{A} + S.$$

From the fundamental formula (4) the value of S is,

$$S = \frac{P\Delta c}{Ar^2}.$$

Now to express Δ in terms of l, consider the case of columns with round ends which deflect into a sinusoid curve, whose equation according to Art. 54 is,

$$y = \Delta \sin \frac{\pi x}{l}.$$

The second derivative of y with respect to x is,

$$\frac{d^2y}{dx^2} = \frac{\Delta\pi^2}{l^2} \sin \frac{\pi x}{l}.$$

For the middle of the column where $x = \frac{1}{2}l$, the curvature hence is,

$$\frac{1}{R} = \frac{d^2y}{dx^2} = \frac{\Delta\pi^2}{l^2}.$$

But the investigation of Art. 33 shows also that $SR = Ec$. Hence,

$$\Delta = \frac{Sl^2}{\pi^2 Ec} = \frac{\left(S_c - \frac{P}{A}\right)l^2}{\pi^2 Ec}.$$

The value of the total unit-stress S_c now is,

$$S_c = \frac{P}{A}\left(1 + \frac{\left(S_c - \frac{P}{A}\right)l^2}{\pi^2 E r^2}\right),$$

and this is the same as (10), except that q has been replaced by $\frac{S_c - \frac{P}{A}}{\pi^2 E}$ which is not a constant since it varies with $\frac{P}{A}$. This expression is a quadratic with reference to $\frac{P}{A}$, and by solution are found the two values,

$$\frac{P}{A} = S_c, \qquad \text{and} \qquad \frac{P}{A} = \frac{\pi^2 E r^2}{l^2},$$

the first of which corresponds to the formula for short blocks and the latter to Euler's formula for columns with round ends.

Prob. 103. Prove that $\Delta c' = r^2$ for a column so deflected that there is no stress on the convex side, c' being the distance from that side to the neutral axis of the cross-section.

CHAPTER VI.

ON TORSION AND ON SHAFTS FOR TRANSMITTING POWER.

Art. 64. The Phenomena of Torsion.

Torsion occurs when applied forces tend to cause a twisting of a body around an axis. Let one end of a horizontal shaft be rigidly fixed and let the free end have a lever *p* attached at right angles to its axis. A weight *P* hung at the end of this lever will twist the shaft so that fibers such as *ab*, which were originally horizontal, assume a spiral form *ad* like the strands of a rope. Radial lines such as *cb* will also have moved through a certain angle *bcd*.

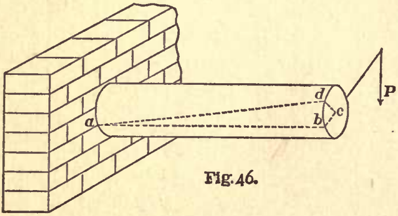


Fig. 46.

Experiments have proved, that if *P* be not so large as to strain the material beyond its elastic limit, the angles *bcd* and *bad* are proportional to *P* and that on the removal of the stress the lines *cd* and *ad* return to their original positions *cb* and *ab*. The angle *bcd* is evidently proportional to the length of the shaft, while *bad* is independent of the length. If the elastic limit be exceeded this proportionality does not hold, and if the twisting be great enough the shaft will be ruptured. These laws are but a particular case of the general axioms stated in Art. 3.

The product Pp is the moment of the force P with respect to the axis of the shaft, p being the perpendicular distance from that axis to the line of direction of P, and is called the twisting moment. Whatever be the number of forces acting at the end of the shaft, their resulting twisting moment may always be represented by a single product Pp.

A graphical representation of the phenomena of torsion may be made as in Fig. 1, the angles of torsion being taken as abscissas and the twisting moments as ordinates. The curve is then a straight line from the origin until the elastic limit of the material is reached, when a rapid change occurs and it soon becomes nearly parallel to the axis of abscissas. The total angle of torsion, like the total ultimate elongation, serves to compare the relative ductility of specimens.

Prob. 104. A shaft 2 feet long is twisted through an angle of 7 degrees by a force of 200 pounds acting at a distance of 6 inches from the center. Through what angle will a shaft 4 feet long be twisted by a force of 500 pounds acting at a distance of 18 inches from the center?

ART. 65. THE FUNDAMENTAL FORMULA FOR TORSION.

The stresses which occur in torsion are those of shearing, each cross-section tending to shear off from the one adjacent to it. When equilibrium obtains the external twisting moment is exactly balanced by the sum of the moments of these internal stresses, or,

$$\text{Resisting moment} = \text{Twisting moment}.$$

If P be the force acting at a distance p from the center about which the twisting takes place, the value of the twisting moment is Pp. To find the resisting moment, let c be the distance from the center to the remotest part of the cross-section where the unit-shear is S_s. Then since the stresses vary as their distances from the center,

$$\frac{S_s}{c} = \text{unit-stress at a unit's distance from center,}$$

$\frac{S_s z}{c}$ = unit-stress at a distance z from center,

$\frac{aS_s z}{c}$ = total stress on an elementary area a,

$\frac{aS_s z^2}{c}$ = moment of this stress,

$\Sigma\frac{aS_s z^2}{c}$ = internal resisting moment.

This may be written $\frac{S_s}{c}\Sigma az^2$. But Σaz^2 is the polar moment of inertia of the cross-section and may be denoted by J. Therefore,

$$(11) \qquad \frac{S_s J}{c} = Pp,$$

which is the fundamental formula for torsion.

The analogy of formula (11) with formula (4) for the flexure of beams will be noted. Pp, the twisting moment, is often the resultant of several forces and might have been expressed by a single letter like the M in (4). By means of (11) a shaft subjected to a given moment may be investigated, or the proper size be determined for a shaft to resist given forces.

Prob. 105. A circular shaft is subjected to a maximum shearing unit-stress of 2 000 pounds when twisted by a force of 90 pounds at a distance of 27 inches from the center. What unit-stress will be produced in the same shaft by two forces of 40 pounds one acting at 21 and the other at 36 inches from the center?

Art. 66. Polar Moments of Inertia.

The polar moment of inertia for simple figures is readily found by the help of the calculus, as explained in works on elementary mechanics. It is also a fudamental principle that,

$$J = I_1 + I_2,$$

where J is the polar moment of inertia, I_1 the least and I_2 the greatest rectangular moment of inertia about two axes passing through the center. The following are values of J for some of the most common cases.

For a circle with a diameter d, $J = \frac{\pi d^4}{32}$.

For a square whose side is d, $J = \frac{d^4}{6}$.

For a rectangle with sides b and d, $J = \frac{bd^3}{12} + \frac{b^3d}{12}$.

The value of c in all cases is the distance from the center about which the twisting occurs, usually the center of figure of the cross-section, to the remotest part of the cross-section. Thus,

For a circle with diameter d, $c = \frac{1}{2}d$,

For a square whose side is d, $c = d\sqrt{\frac{1}{2}}$.

For a rectangle with sides b and d, $c = \frac{1}{2}\sqrt{b^2 + d^2}$.

It is rare in practice that formulas for torsion are needed for any cross-sections except squares and circles.

Prob. 106. Find the values of J and c for an equilateral triangle whose side is d.

ART. 67. THE CONSTANTS OF TORSION.

The constant S_s computed from experiments on the rupture of shafts by means of formula (11) may be called the modulus of torsion, in analogy with the modulus of rupture as computed from (4). As would be expected the values thus found agree closely with the ultimate shearing unit-stress given in Art. 7, viz.,

For timber, $S_s =$ 2 000 pounds per square inch,
For cast iron, $S_s =$ 25 000 pounds per square inch,
For wrought iron, $S_s =$ 50 000 pounds per square inch,
For steel, $S_s =$ 75 000 pounds per square inch.

By the use of these average values it is hence easy to compute from (11) the load P acting at the distance p which will cause the rupture of a given shaft.

The coefficient of elasticity for shearing may be computed from experiments on torsion in the following manner. Let a circular shaft whose length is l and diameter d be twisted through an angle θ by the twisting moment Pp. Here a point on the circumference of one end is twisted relative to a corresponding point on the other end through the arc θ or through the distance $\frac{1}{2}\theta d$. From the fundamental definition of the coefficient of elasticity E as given in (1),

$$E = \frac{S_s}{s} = \frac{2S_s l}{\theta d},$$

and inserting for S_s its value from (11),

$$E = \frac{32Ppl}{\pi \theta d^4},$$

from which E can be computed when all the quantities in the second member have been determined by experiment, provided that the elastic limit of the material be not exceeded.

Prob. 107. An iron shaft 5 feet long and 2 inches in diameter is twisted through an angle of 7 degrees by a force of 5 000 pounds acting at 6 inches from the center, and on the removal of the force springs back to its original position. Find the value of E for shearing.

Prob. 108. What force P acting at the end of a lever 24 inches long will twist asunder a steel shaft 1.4 inches in diameter?

Art. 68. Shafts for the Transmission of Horse Power.

Work is the product of a resistance by the distance through which it acts, and is usually measured in foot-pounds. A horse-power is 33 000 foot-pounds of work done in one minute. It is required to determine the relation between the horse-power H transmitted by a shaft and the greatest internal shearing unit-stress S_s produced in it.

Let a shaft making n revolutions per minute transmit H horse-power. The work may be applied by a belt from the motor to a pulley on the shaft, then, by virtue of the elasticity and resistance of the material of the shaft, it is carried through other pulleys and belts to the working machines. In doing this the shaft is strained and twisted, and evidently S_s increases with H. Let P be the resistance acting at the circumference of the pulley and p the radius of the pulley. In making one revolution the force P acts through the distance $2\pi p$ and performs the work $2\pi pP$, and in n revolutions it performs the work $2\pi pPn$. Then if P be in pounds and p in inches the imparted horse-power is,

$$H = \frac{2\pi pPn}{33\,000 \times 12}.$$

The twisting moment Pp in this expression may be expressed, as in formula (11), by the resisting moment $\frac{S_s J}{c}$. Hence the equation becomes,

$$(12) \qquad H = \frac{\pi n S_s J}{198\,000c}.$$

This is the formula for the discussion of shafts for the transmission of power, and in it J and c must be taken in inches and S_s in pounds per square inch, while n is the number of revolutions per minute.

Prob. 109. What horse-power is required to draw 25 miles per hour a train weighing 400 tons on a track where the coefficient of friction is 0.006 and the grade 30 feet per mile?

Prob. 110. A wooden shaft 6 inches square breaks when making 40 revolutions per minute. Find the horse-power then probably transmitted.

ART. 69. ROUND SHAFTS.

For round shafts of diameter d, the values of J and c are to be taken from Art. 66 and inserted in the last equation, giving,

$$S_s = 321\,000\frac{H}{nd^3}, \qquad \text{or} \qquad d = 68.5\sqrt[3]{\frac{H}{nS_s}}.$$

The first of these may be used for investigating the strength of a given shaft when transmitting a certain number of horse-power with a known velocity. The computed values of S_s, compared with the ultimate values in Art. 67, will indicate the degree of security of the shaft. Here d must be taken in inches and S_s will be in pounds per square inch.

The second equation may be used for determining the diameter of a shaft to transmit a given horse-power with a given number of revolutions per minute. Here a safe allowable value must be assumed for S_s in pounds per square inch, and then d will be found in inches. This equation shows that the diameter of a shaft varies directly as the cube root of the transmitted horse-power and inversely as the cube root of its velocity.

Prob. 111. Find the factors of safety for a wrought iron shaft 2½ inches in diameter when transmitting 25 horse-powers while making 100 revolutions per minute, and also when making 10 revolutions per minute.

Prob. 112. Find the diameter of a wrought iron shaft to transmit 90 horse-powers with a factor of safety of 8 when making 250 revolutions per minute, and also when making 100 revolutions per minute.

Art. 70. Square Shafts.

For a square shaft whose side is d formula (12) reduces to,

$$S_s = 267\,500 \frac{H}{nd^3}, \qquad \text{or} \qquad d = 64.4 \sqrt[3]{\frac{H}{nS_s}}.$$

These are the same as for round shafts except in the numerical constants, and are to be used in the same manner, the first to investigate an existing shaft and the second to find the diameter for one proposed.

Prob. 113. Find the proper diameter of a wooden shaft for a water wheel which is to transmit 8 horse-power at 20 revolutions per minute.

Prob. 114. Find the factor of safety of a wooden shaft 12 inches square when transmitting 16 horse-power at 40 revolutions per minute.

ART. 71. MISCELLANEOUS EXERCISES.

Exercise 7. Find in the library a description of Thurston's autographic testing machine for torsion. Write a condensed description of it and of the method of its use. Give sketches of the autographic recording apparatus. Explain how the torsion diagrams may be used to study the relative stiffness, elastic limit and ductility of specimens.

Exercise 8. Go to a testing room and inspect Thurston's testing machine for torsion. Ascertain the dimensions and kind of specimens tested thereon. Explain with sketches the construction of the machine and the method of its use. State how the quality of the specimens is inferred from the torsion diagrams.

Exercise 9. Measure the diameter of a shaft, and ascertain its velocity and the number of transmitted horse-powers. State in a short report the data and the results of your investigations.

Prob. 115. Compare the strength of a square shaft with that of a circular shaft of equal areas.

Prob. 116. Jones & Laughlins give the formulas,

$$d = \sqrt[3]{\frac{62.5H}{n}}, \quad \text{and} \quad d = \sqrt[3]{\frac{37.5H}{n}},$$

the first for ordinary turned wrought iron shafts, and the second for cold rolled wrought iron shafts. What working unit-stresses do these imply?

CHAPTER VII.

ON COMBINED STRESSES.

ART. 72. CASES OF COMBINED STRESSES.

The three kinds of simple stress are tension, compression and shear, or, in other words, the numerical investigation of bodies under stress includes only the unit-stresses S_t, S_c and S_s. Transverse or flexural stress was investigated in Chapter III by resolving the internal stresses into tension, compression and shear. Torsional stress is merely a particular case of shear.

Tension and compression are similar in character and differ only in sign or direction. Hence their combination is effected by algebraic addition. Thus if P be a tensile stress and P' a compressive stress applied to the same bar at the same time the resultant stress is $P - P'$ which may be either tensile or compressive.

Tension and shear, or compression and shear, are often combined, as internally in the case of beams and externally under many circumstances.

Tension and flexure are combined when loads are placed upon a bar under tension. This case and that of compression and flexure are of frequent occurence, and their investigation is of much practical importance.

Flexure and torsion are combined whenever shafts for the transmission of power are loaded with pulleys and belts, and, as will be seen, the effect of the flexure is sensibly to modify the formulas of the last chapter. Compression and flexure occur in the case of vertical shafts.

The internal stresses in a body produced by applied forces are usually of a complex character. Even in a case of simple tension there are shearing stresses in all directions except those perpendicular and parallel to the line of tension. If P be the tensile force and A the area of the cross-section of the bar the tensile unit-stress is $\frac{P}{A}$, and it may be shown, as in Prob. 26, that a shearing unit-stress of $\frac{1}{2}\frac{P}{A}$ exists in a section making an angle of 45 degrees with the axis of the bar.

Prob. 117. A pulls 29 pounds at one end of a rope and B pulls 30 pounds at the other end. What is the stress in the rope?

ART. 73. STRESSES DUE TO TEMPERATURE.

If a bar be unstrained it expands when the temperature rises and contracts when the temperature falls. But if the bar be under stress, so that the change of length cannot occur, an additional unit-stress must be produced which will be equivalent to the unit-stress that would cause the same change of length in the unstrained bar. Thus if a rise of temperature elongates a bar of length unity the amount s when free from stress, it will cause the unit-stress $S = sE$ (see Art. 4) when the bar is prevented from expanding by external forces.

Let l be the length of the bar, a its coefficient of linear expansion for a change of one degree, and λ the change of length due to the rise or fall of t degrees. Then,

$$\lambda = atl,$$

and the unit-strain s is,

$$s = \frac{\lambda}{l} = at.$$

The unit-stress produced by the change in temperature hence is,

$$S = atE$$

which is seen to be independent of the length of the bar. The total stress on the bar is then AS.

The following are average values of the coefficients of linear expansion for a change in temperature of one degree Fahrenheit.

For brick and stone,	$a = 0.000\,00\,50$,
For cast iron,	$a = 0.000\,00\,62$,
For wrought iron,	$a = 0.000\,00\,67$,
For steel.	$a = 0.000\,00\,65$.

As an example consider a wrought iron tie rod 20 feet in length and 2 inches in diameter which is screwed up to a tension of 9 000 pounds in order to tie together two walls of a building. Let it be required to find the stress in the rod when the temperature falls 10° F. Here,

$$S = 0.000\,00\,67 \times 10 \times 25\,000\,000 = 1\,675 \text{ pounds.}$$

The total tension in the rod now is,

$$9\,000 + 3.14 \times 1\,675 = 14\,000 \text{ pounds.}$$

Should the temperature rise 10° the tension in the rod would be,

$$9\,000 - 3.14 \times 1\,675 = 4\,000 \text{ pounds.}$$

In all cases the stresses caused by temperature are added or substracted to the tensile or compressive stresses already existing.

Prob. 118. A cast iron bar is confined between two immovable walls. What unit-stress will be produced by a rise of 40° in temperature?

Art. 74. Combined Tension and Flexure.

Consider a beam in which the flexure produces a unit-stress S at the fiber on the tensile side most remote from the neutral axis. Let a tensile stress P be then applied to the ends of the bar uniformly distributed over the cross-section A. The tensile unit-stress at the neutral surface is then $\frac{P}{A}$ and all the longitudinal stresses due to the flexure are increased by this amount. The maximum tensile unit-stress is then $\frac{P}{A} + S$ in which S is to be found from formula (4).

In designing a beam under combined tension and flexure the dimensions must be so chosen that $\frac{P}{A} + S$ shall not exceed the proper allowable working unit-stress. For instance, let it be required to find the size of a square wooden beam of 12 feet span to hold a load of 300 pounds at the middle while under a longitudinal stress of 2 000 pounds, so that the maximum tensile unit-stress may be about 1 000 pounds per square inch. Let d be the side of the square. From formula (4),

$$S = \frac{6M}{d^3} = \frac{6 \times 150 \times 72}{d^3}.$$

Then from the conditions of the problem,

$$\frac{2\,000}{d^2} + \frac{64\,800}{d^3} = 1\,000,$$

from which results the cubic equation,

$$d^3 - 2d = 64.8,$$

whose solution gives for d the value 4.25 inches.

In investigating a beam under combined tension and flexure the maximum value of $\frac{P}{A} + S$ is to be computed, and the factor of safety found by comparing it with the ultimate tensile strength of the material.

Prob. 119. A heavy 12-inch **I** beam carries a uniform load of 200 pounds per linear foot, besides its own weight, and is subjected to a longitudinal tension of 80 000 pounds. Find the factor of safety of the beam.

Prob. 120. What **I** beam is required to carry a uniform load of 200 pounds per linear foot when subjected to a tension of 50 000 pounds, the maximum tensile stress to be 9 000 pounds per square inch?

ART. 75. COMBINED COMPRESSION AND FLEXURE.

Consider a beam in which the flexure produces a unit-stress S in the fiber on the compressive side most remote from the neutral axis. Let a compressive stress P be applied in the direction of its length uniformly over the cross-section A. Then at the neutral surface the unit-stress is $\frac{P}{A}$ and at the remotest fiber it is $\frac{P}{A} + S$. The discussion of this case is hence exactly similar to that of the last article. If the beam is short the total working unit-stress is to be taken as for a short prism; if long it should be derived from Gordon's formula for columns.

The method of investigation explained in this and the preceding article is the one ordinarily used in practice on account of the complexity of the formulas which result from the strict mathematical determination of the moments of the applied forces. Although not exact the method closely approximates to the truth, giving values of the stresses a little too large for the case of tension and a little too small for the case of compression.

An inclined beam is an instance of combined flexure and compression. In the case shown in Fig. 47 the reactions are vertical and their values for any given loads are found by the principles of Art. 14. Let φ be the inclination of the beam to the vertical, and for illustration let the load be uniform. Then $R_1 = R_2 = \frac{1}{2}wl$, if w is the load per linear unit.

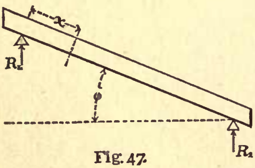


Fig. 47.

At any section x the unit-stress S due to the flexure is added to the compressive unit-stress S_c due to the components of R_1 and wx which are parallel to the beam. Thus for a rectangular beam whose breadth is b and depth d the formula (4) gives,

$$S = \frac{6M}{bd^2} = \frac{3wlx \cos \varphi - 3wx^2 \cos \varphi}{bd^2},$$

while from (1) the direct compression is,

$$S_c = \frac{wx \sin \varphi - R_1 \sin \varphi}{bd},$$

The total compressive unit-stress at the fiber on the upper side now is,

$$S_x = S + S_c = \frac{3w \cos \varphi}{bd^2}(lx - x^2) + \frac{w \sin \varphi}{2bd}(2x - l).$$

It is easy to show that this is a maximum when,

$$x = \frac{l}{2} + \frac{d \tan \varphi}{6},$$

and therefore the maximum unit-stress is,

$$S_m = \frac{3wl^2 \cos \varphi}{4bd^2} + \frac{w \sin \varphi \tan \varphi}{4b}.$$

This equation may be used to determine the factor of safety of a given beam or to design one proposed.

A rafter of a roof is a case where one of the reactions of the inclined beam is horizontal as shown in Fig. 48. If l be the length, w the load per linear unit and φ the inclination, H is to be found by Art. 14; thus taking the center of moments at the lower end,

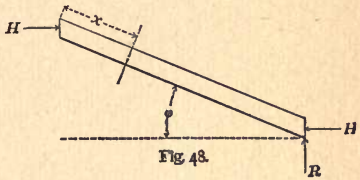


Fig. 48.

$$H \,.\, l \sin \varphi = wl \,.\, \frac{l \cos \varphi}{2}, \quad \text{whence} \quad H = \frac{wl}{2} \cot \varphi,$$

For any section x, the flexural unit-stress now is,

$$S = \frac{6(Hx \sin \varphi - \frac{1}{2}wx^2 \cos \varphi)}{bd^2},$$

and the uniform compressive unit-stress is,

$$S_c = \frac{H \cos \varphi + wx \sin \varphi}{bd}.$$

The total compressive unit-stress on the upper fiber hence is,

$$S_x = S_c + S = \frac{3w \cos \varphi}{bd^2}(lx - x^2) + \frac{wl \cot \varphi \cos \varphi}{2bd} + \frac{wx \sin \varphi}{bd}.$$

This is a maximum when x has the same value as for the last case, and,

$$S_m = \frac{3wl^2 \cos\varphi}{4bd^2} + \frac{wl \operatorname{cosec}\varphi}{2bd} + \frac{w \sin\varphi \tan\varphi}{b}$$

is the greatest compressive unit-stress.

In any inclined rafter let P denote all the load above a section distant x from the upper end. Then the greatest unit-stress for that section is,

$$S_w = \frac{Mc}{I} + \frac{P \sin\varphi}{A} + \frac{H \cos\varphi}{A},$$

from which S_w may be found for any given case.

Prob. 121. A wooden beam 10 inches wide and 8 feet long carries a uniform load of 500 pounds per linear foot and is subjected to a longitudinal compression of 40 000 pounds. Find the depth of the beam so that the maximum working unit-stress may be about 800 pounds per square inch.

Prob. 122. A roof with two equal rafters is 40 feet in span and 15 feet in height. The wooden rafters are 4 inches wide, 6 inches deep and carry 450 pounds per linear foot. Find the factor of safety of the rafters.

Prob. 123. A roof with two equal rafters is 40 feet in span and 15 feet in height. The wooden rafters are 4 inches wide and each carries a load of 450 pounds at the center. Find the depth of the rafter so that S_m may be 700 pounds per square inch.

Art. 76. Shear Combined with Tension or Compression.

Let a bar whose cross-section is A be subjected to the longitudinal tension or compression P and at the same time to a shear V at right angles to its length. The longitudinal unit-stress is $\frac{P}{A}$ which may be denoted by p, and the shearing unit-stress is $\frac{V}{A}$ which may be denoted by v. It is required to find the maximum unit-stresses produced by the combination of p and v. In the following demonstration P will be regarded as a tensile force, although the reasoning and conclusions apply equally well when it is compressive.

Consider an elementary cubic particle with edges one unit in length acted upon by the horizontal tensile force p and p, and by the vertical shear v and v, as shown in Fig. 49. These forces are not in equilibrium unless a horizontal couple be applied as in the figure, each of whose forces is equal to v. Therefore at every point of a body under vertical shear there exists a horizontal shear, and the horizontal shearing unit-stress is equal to the vertical shearing unit-stress.

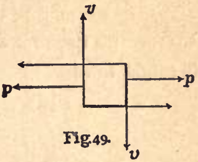


Fig. 49.

Let a parallelopipedal element have the length dm, the height dn and a width of unity. The tensile force pdn tends to pull it apart longitudinally. The vertical shear vdn tends to cause rotation and this is resisted, as shown above, by the horizontal shear vdm. These forces may be resolved into rectangular components parallel and perpendicular to the diagonal dz, as shown in Fig. 50. The components parallel to the diagonal form a shearing force sdz, and those perpendicular to it a tensile force tdz, s being the shearing and t the tensile unit-stresses. Let φ be the angle between dz and dm. The problem is first to state expressions for sdz and tdz in terms of φ, and then to determine the value of φ, or the ratio of dm to dn, which gives the maximum values of s and t.

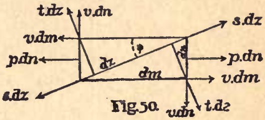


Fig. 50.

By simple resolution of forces,

$$sdz = pdn \cos \varphi + vdm \cos \varphi - vdn \sin \varphi,$$
$$tdz = pdn \sin \varphi + vdm \sin \varphi + vdn \cos \varphi.$$

Divide each of these by dz, for $\frac{dn}{dz}$ put its value $\sin \varphi$ and for $\frac{dm}{dz}$

its value cos φ. Then the equations take the form,

$$s = p \sin\varphi \cos\varphi + v(\cos^2\varphi - \sin^2\varphi),$$
$$t = p \sin^2\varphi + 2v \sin\varphi \cos\varphi.$$

These may be written,

$$s = \tfrac{1}{2}t \sin 2\varphi + s \cos 2\varphi,$$
$$t = \tfrac{1}{2}t\,(1 - \cos 2\varphi) + s \sin 2\varphi.$$

By placing the first derivative of each of these equal to zero it is found that,

$$s \text{ is a maximum when } \tan 2\varphi = \frac{p}{2v},$$

$$t \text{ is a maximum when } \tan 2\varphi = -\frac{2v}{p}.$$

Expressing sin 2φ and cos 2φ in terms of tan 2φ and inserting them in the above the following values result,

$$(13) \qquad \begin{cases} \max s = \pm\sqrt{v^2 + \dfrac{p^2}{4}} \\ \max t = \dfrac{p}{2} + \sqrt{v^2 + \dfrac{p^2}{4}}. \end{cases}$$

These formulas apply to the discussion of the internal stresses in beams, as well as to combined longitudinal stress and vertical shear directly applied by external forces. If p is tension t is tension; if p is compression t is also compression.

Prob. 124. A rivet ¾-inch in diameter is subjected to a tension of 2 000 pounds and at the same time to a cross shear of 3 000 pounds. Find the maximum tensile and shearing unit-stresses and the directions they make with the axis of the rivet.

Art. 77. Combined Flexure and Torsion.

This case occurs when a shaft for the transmission of power is loaded with weights. Let S be the greatest flexural unit-stress computed from (4) and S_s the torsional shearing unit-stress computed from (12) or by the special equations of Arts. 68 and 69.

Then, according to the last article the resultant maximum unit-stresses are,

$$\text{max. ten. or comp.}\quad t = \frac{S}{2} + \sqrt{S_s^2 + \frac{S^2}{4}}$$

$$\text{max shear}\qquad s = \pm\sqrt{S_s^2 + \frac{S^2}{4}}.$$

For wrought iron or steel it is usually necessary to regard only the first of these unit-stresses, but for timber the second should also be kept in view.

For example, let it be required to find the factor of safety of a wrought iron shaft 3 inches in diameter and 12 feet between bearings, which transmits 40 horse-power while making 120 revolutions per minute, and upon which a load of 800 pounds is brought by a belt and pulley at the middle. Taking the shaft as fixed over the bearings the flexural unit-stress is,

$$S = \frac{4Pl}{\pi d^3} = 5\,400 \text{ pounds per square inch.}$$

From Art. 68 the torsional unit-stress is,

$$S_s = 321\,000\frac{H}{nd^3} = 4\,000 \text{ pounds per square inch.}$$

The maximum tensile and compressive unit-stress now is,

$$t = 2\,700 + \sqrt{4\,000^2 + 2\,700^2} = 7\,600 \text{ pounds per square in.}$$

and the factor of safety is hence over 7.

As a second example, let it be required to find the size of a square wooden shaft for a water wheel weighing 3 000 pounds which transmits 8 horse-power while making 20 revolutions per minute. The length of the shaft is 16 feet and one-third of the weight is concentrated at the center and the remainder is equally divided between two points each 6 feet from the center. Here the greatest flexural unit-stress is,

$$S = \frac{6(1\,500 \times 96 - 1\,000 \times 72)}{d^3} = \frac{49\,200}{d^3},$$

and from Art. 69 the torsional unit-stress is,

$$S_s = \frac{267\,500 \times 8}{20d^3} = \frac{107\,000}{d^3}.$$

From the formula of the last article the combined tensile or compressive stress is,

$$t = \frac{134\,400}{d^3}.$$

Now if the working value of t be taken at 600 pounds per square inch the value of d will be about 6 inches. From formula (13) also

$$s = \frac{109\,800}{d^3},$$

and if the working value of s be taken at 150, the value of d is found to be about 9 inches. The latter value should hence be chosen for the size of the shaft.

Prob. 125. Prove that the formula for finding the diameter of a round iron shaft is,

$$d^3 = \frac{16M}{\pi t} + \frac{16}{t}\sqrt{\frac{M^2}{\pi^2} + \frac{402\,500\,000\,H^2}{n}},$$

where M is the maximum bending moment of the transverse forces in pound-inches, H the number of transmitted horse-power, n the number of revolutions per minute, and t the safe allowable tensile or compressive working strength of the material.

ART. 78. COMBINED COMPRESSION AND TORSION,

In the case of a vertical shaft the torsional unit-stress S_s combines with the direct compressive stress due to the weights upon the shaft, and produces a resultant compression t and shear s. From formulas (13) the combined stresses are,

$$t = \frac{S_c}{2} + \sqrt{\frac{S_c^2}{4} + S_s^2}$$

$$s = \sqrt{\frac{S_c^2}{4} + S_s^2}$$

The use of these is the same as those of the last article, S_s being found from the formulas of chapter VI, while S_c is computed from formula (1) if the length of the shaft be less than ten times its diameter and from (10) for greater lengths.

Prob. 126. A vertical shaft, weighing with its loads 6 000 pounds, is subjected to a twisting moment by a force of 300 pounds acting at a distance of 4 feet from its center. If the shaft is wrought iron, 4 feet long and 2 inches in diameter, find its factor of safety.

Prob. 127. Find the diameter of a short vertical steel shaft to carry loads amounting to 6 000 pounds when twisted by a force of 300 pounds acting at a distance of 4 feet from the center, taking the unit-stress against tension as 10 000 and against shearing as 7 000 pounds per square inch.

CHAPTER VIII.

APPENDIX AND TABLES.

ART. 79. HORIZONTAL SHEAR IN BEAMS.

The common theory of flexure as presented in Chapters III and IV considers that the internal stresses at any section are resolved into their horizontal and vertical components, the former producing longitudinal tension and compression and the latter a transverse shear, and that these act independently of each other. Formula (3) supposes further that the vertical shear is uniformly distributed over the cross-section of the beam. A closer analysis will show that a horizontal shear exists also and that this, together with the vertical shear, varies in intensity from the neutral surface to the upper and lower sides of the beam. It is well-known that a pile of boards which acts like a beam deflects more than a solid timber of the same depth, and this is largely due to the lack of horizontal resistance between the layers. The common theory of flexure in neglecting the horizontal shear generally errs on the side of safety. In a few experiments however beams have been known to crack along the neutral surface and it is hence desirable to investigate the effect of horizontal shear in tending to cause rupture of that kind. That a horizontal shear exists simultaneously with the vertical shear is evident from the considerations in Art. 76.

Let Fig. 51 represent a portion of a bent beam of uniform section. Let a rectangular notch *nmpq* be imagined to be cut into it, and let forces be applied to it to preserve the equilibrium. Let H be the sum of all the horizontal components of these forces acting on mn and H' the sum of those acting on qp. Now

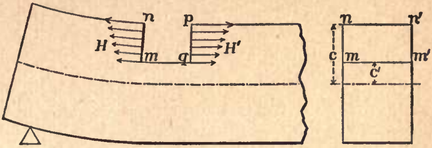


Fig 51.

H' is greater or less than H, hence the difference $H' - H$ must act along mq as a horizontal shear. Let the distance mq be dx, the thickness mm' be b, and the area $mqmm'$ be at a distance c' above the neutral surface. Let c be the distance from that neutral surface to the remotest fiber where the unit-stress is S. Let a be the cross-section of any fiber. Let M be the bending moment at the section mn and M' that at the section qp. Now from the fundamental laws of flexure,

$\frac{S}{c}$ = unit-stress at a unit's distance from neutral surface.

$\frac{S}{c}y$ = unit-stress at distance y from neutral surface,

$\frac{aSy}{c}$ = total stress on fiber a at distance y,

$\frac{S}{c}\Sigma_{c'}^{c}ay$ = sum of horizontal stresses between m and n.

The value of H hence is, since $\frac{S}{c} = \frac{M}{I}$,

$$H = \frac{M}{I}\Sigma_{c'}^{c}ay,$$

and likewise,

$$H' = \frac{M'}{I}\Sigma_{c'}^{c}ay.$$

The horizontal shear therefore is,

$$H' - H = \frac{M' - M}{I}\Sigma_{c'}^{c}ay.$$

Now since the distance mq is dx, the value of $M' - M$ is dM. Also if S_h be the horizontal shearing unit-stress upon the area bdx the value of $H' - H$ is $S_h bdx$. Hence,

$$S_h = \frac{dM}{Ibdx} \Sigma_{c'}^{c} ay.$$

Again from Art. 41 it is plain that $\frac{dM}{dx}$ is the vertical shear V at the section under consideration. Therefore,

$$(\text{⊦}1) \qquad S_h = \frac{V}{Ib} \Sigma_{c'}^{c} ay.$$

is the formula for the horizontal unit-shear at any point of the beam.

This expression shows that the horizontal unit-shear is greatest at the supports, and zero at the dangerous section where V is zero. The summation expression is the statical moment of the area $mm'nn'$ with reference to the neutral axis; it is zero when $y = c$, and a maximum when $y = 0$. Hence the longitudinal unit-shear S_h is zero at the upper and lower sides of the beam and is a maximum at the neutral surface. The formula for the maximum horizontal unit-shear therefore is,

$$S_h = \frac{\max V}{Ib} \Sigma_{o}^{c} ay.$$

Here I is the moment of inertia of the whole cross-section with reference to the neutral axis (Art. 22), b is the width of the beam along the neutral surface, and $\Sigma_{o}^{c} ay$ is the statical moment of the area of the part of the cross-section on one side of the neutral axis.

For a rectangular beam of breadth b and depth d, the value of I is $\frac{bd^3}{12}$, and $\Sigma_{o}^{c} ay = \frac{bd}{2} \cdot \frac{d}{4} = \frac{bd^2}{8}$. Then,

$$S_h = \frac{3V}{2bd}.$$

By inserting in this the values of V for particular sections the corresponding values of S_h are found.

Prob. 128. Prove that for a cylindrical beam of diameter d the horizontal unit-shear along the neutral surface is $\frac{32V}{3\pi d^2}$.

Prob. 129. In the Journal of the Franklin Institute for February, 1883, is detailed an experiment on a spruce joist 3⅞ × 12 inches and 14 feet long, which broke by tension at the middle and afterwards by shearing along the neutral axis at the end when loaded at the middle with 12 545 pounds. Find the tensile and shearing unit-stresses.

ART. 80. MAXIMUM INTERNAL STRESSES IN BEAMS.

From the last article it is evident that at every point of a beam there exists a horizontal unit-shear of the intensity S_h and also a vertical unit-shear of the same intensity, whose value is given by (14). At every point there also exists a longitudinal tension or compression which may be computed from (4) with the aid of the principle that these stresses vary directly as their distances from the neutral axis. Let v denote the unit-shear thus determined and p the tensile or compressive unit-stress. Then from Art. 76 the maximum unit-shear at that point is,

$$s = \sqrt{v^2 + \frac{p^2}{4}},$$

and it makes an angle φ with the neutral surface such that,

$$\tan 2\varphi = \frac{p}{2v}.$$

Also the tensile or compressive unit-stress at that point is,

$$t = \frac{p}{2} + \sqrt{v^2 + \frac{p^2}{4}},$$

and it makes an angle θ with the neutral surface such that,

$$\tan 2\theta = -\frac{2v}{p}.$$

From these formulas the lines of direction of the maximum stresses may be traced throughout the beam.

For the maximum shear v is greatest and p is zero at the neutral surface, while v is zero and p is greatest at the upper and lower surfaces. Hence for the neutral surface φ is 0, it increases with p, and becomes $45°$ at the upper and lower surfaces.

For the maximum tension t is greatest and equal to p on the convex side where $v = 0$ and $\theta = 0$. As the neutral surface is approached v increases, p decreases, and θ increases. At the neutral surface v is greatest, p is zero, and $\theta = -45°$. Here the maximum tension and compression are each equal to v.

For the maximum compression in like manner θ is $0°$ at the concave surface and $45°$ at the neutral surface. The lines of maximum tension if produced beyond the neutral surface would evidently cut those of maximum compression at right angles and be vertical at the concave surface.

The following figure is an attempt to represent the lines of maximum stress in a beam. The full lines above the neutral surface are those of maximum compression, while those below are maximum tension. The broken lines are those of maximum shear. On any line the intensity of stress varies with the inclination, being greatest where the line is horizontal and least where its inclination is $45°$. The lines of maximum shear cut those of maximum tension and compression at angles of $45°$. The lines of maximum tension above the neutral surface and those of maximum compression below it are not shown.

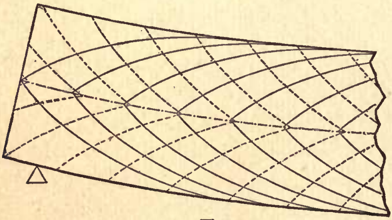


Fig. 52.

It appears from the investigation that the common theory of flexure gives the horizontal unit-stress correctly at the dangerous section of a simple beam where the vertical shear is zero. At other sections the stress S as computed from (4) is correct for the remotest fiber, but for other fibers the unit-stress t is greater. It is hence seen that the main practical value of the theory of internal stress is in showing that the intensity of the shear varies throughout the cross-section of the beam. For a restrained beam, where the vertical shear suddenly changes sign at the dangerous section, the common theory gives the horizontal stress S correctly for the remotest fiber only, and it might be possible for the maximum stress t to be greater than S for a fiber nearer to the neutral surface.

Prob. 130. A joist fixed at both ends is 3 × 12 inches and 12 feet long, and is strained by a load at the middle, so that the value of S as computed from (4) is 4 000 pounds per square inch. Find the value of t for points over the support distant 3, 4 and 5 inches from the neutral surface.

ART. 81. THE FATIGUE OF MATERIALS.

The ultimate strength S_u is usually understood to be that unit-stress which causes rupture at one application. Experience and experiments, however, teach that if a unit-stress somewhat less than S_u be applied a sufficient number of times to a bar rupture will be caused. The experiments of Wöhler have been of the greatest value in establishing the laws which govern the rupture of metals under repeated applications of stress. For instance, he found that the rupture of a bar of wrought iron by tension was caused in the following different ways.

By 800 applications of 52 800 pounds per square inch.
By 107 000 applications of 48 400 pounds per square inch.
By 450 000 applications of 39 000 pounds per square inch.
By 10 140 000 applications of 35 000 pounds per square inch.

The range of stress in each of these applications was from 0 to the designated number of pounds per square inch. Here it is seen that the breaking stress decreases as the number of applications increase. In other experiments where the initial stress was not 0, but a permanent value S, the same law was seen to hold good. It was further observed that a bar could be strained from 0 up to its elastic limit an enormous number of times without rupture. From a discussion of his numerous experiments Wöhler stated the following laws.

1. By repeated applications of stress rupture may be caused by a unit-stress less in value than the ultimate strength of the material.
2. The greater the range of stress the less is the unit-stress required to produce rupture after an enormous number of applications.
3. When the stress ranges from 0 up to a value about equal to the elastic limit the number of applications required to rupture it is enormous, or greater than could be applied in practice.
4. A range of stress from tension into compression, or *vice versa*, produces rupture sooner than the same range in stress of one kind only.
5. When the range of stress in tension is equal to that in compression the stress which will produce rupture after an enormous number of applications is a little greater than one-half the elastic limit.

The term 'enormous number' used in stating these laws means about 40 millions, that being roughly the number used by Wöhler to cause rupture under the conditions stated. For all practical cases of repeated stress, except in fast moving machinery, this great number would seldom be exceeded during the natural life of the piece.

In Art. 8 it was recognized that the working strength should be less for pieces subject to varying stresses than for those carrying steady loads only. For many years indeed it has been the practice of designers to grade the working strength according to the range of stresses to which it might be liable to be subjected, Wöhler's laws and experiments afford however a means of grading these values in a more satisfactory manner than mere judgment can do, and a formula for that purpose will be deduced in the next article. After the working strength S_w is determined the cross-section of the piece is found in the usual way, if in tension by formula (1), and if in compression by formula (1) or (10) as the case may require.

Prob. 131. How many years will probably be required for a tie bar in a bridge truss to receive 40 million repetitions of stress?

ART. 82. WORKING STRENGTH FOR REPEATED STRESSES.

Consider a bar in which the unit-stress varies from S' to S, the latter being the greater numerically. Both S' and S may be tension or both may be compression, or one may be tension and the other compression. In the last case the sign of S' is to be taken as minus. Consider the stress to be repeated an enormous number of times and rupture to then occur. By Wöhler's second law S is some function of the range of stress, or,

$$S = \psi(S - S').$$

This may be expressed in another way, thus,

$$S = \varphi\left(1 - \frac{S'}{S}\right).$$

or, in words, the rupturing stress S after an enormous number of repetitions is a function of the ratio of the limiting stresses.

Let u be the ultimate strength of the material, tensile if S is tension and compressive if S is compression. Let e be the unit-stress at the elastic limit, and f the unit-stress which produces rupture after an enormous number of repetitions when the range of stress in tension is equal to that in compression. It is required to find the value of S in terms of u, e, f and the ratio $\frac{S'}{S}$. For this purpose assume the function,

$$S = me + n\frac{S'}{S} + p\left(\frac{S'}{S}\right)^2$$

in which m, n and p are quantities to be determined. Now if $S' = S$, there is no range of stress, and the case corresponds to that of a steady load for which $S = u$. Again, let $S' = 0$, then by Wöhler's third law $S = e$. Lastly, let $S' = -S$, then by Wöhler's fifth law the value of S is f. For these three conditions the assumed function becomes,

$$\text{For } S' = S, \qquad u = me + n + p,$$
$$\text{For } S' = 0, \qquad e = me,$$
$$\text{For } S' = -S, \qquad f = me - n + p.$$

From these three equations are found the values,

$$m = 1, \qquad n = \frac{u - f}{2}, \qquad p = \frac{u + f - 2e}{2},$$

and the expression for S hence is,

$$(15) \qquad S = e + \frac{u - f}{2} \cdot \frac{S'}{S} + \frac{u + f - 2e}{2}\left(\frac{S'}{S}\right)^2.$$

This formula is not to be regarded as the true law of rupturing strength under repeated stresses, but merely as an empirical statement which agrees with the limiting values determined by experiment, and which will give approximately intermediate values.

In designing a bar which is to be subject to an enormous number of repetitions of stress, ranging from P' to P, the ratio $\frac{P'}{P}$ is the same as $\frac{S'}{S}$, and formula (15) gives the unit-stress S which will cause rupture. To be sure of safety a factor of security must be applied; if a be this factor, (15) becomes the formula for working strength, or,

$$(15)' \qquad S_w = \frac{e}{a}\left(1 + \frac{u - f}{2e}\frac{P'}{P} + \frac{u + f - 2e}{2e}\frac{P'^2}{P^2}\right)$$

from which the proper unit-stress may be computed. The factor of security a is here usually taken the same as the factor of safety for a steady load where there is no range of stress.

For example, consider a kind of wrought iron for which $u = 55\ 000$, $e = 25\ 000$, and $f = 12\ 500$. Then with a factor of 4, formula (15) becomes,

$$S_w = 6\ 250\left(1 + \tfrac{17}{20}\frac{P'}{P} + \tfrac{7}{20}\frac{P'^2}{P^2}\right)$$

Here P' is the minimum and P the maximum stress to which the piece is to be subjected, the first due perhaps to dead load and the second to combined dead and live load. The following are values of S_w for certain ratios of P' to P.

For $\frac{P'}{P} = 1$,	$S_w = 13\ 750$,
For $\frac{P'}{P} = \frac{3}{4}$,	$S_w = 11\ 460$,
For $\frac{P'}{P} = \frac{1}{2}$,	$S_w = 9\ 400$,
For $\frac{P'}{P} = \frac{1}{4}$,	$S_w = 7\ 715$,
For $\frac{P'}{P} = 0$,	$S_w = 6\ 250$,
For $\frac{P'}{P} = -\frac{1}{3}$,	$S_w = 4\ 720$,
For $\frac{P'}{P} = -\frac{2}{3}$,	$S_w = 4\ 030$,
For $\frac{P'}{P} = -1$,	$S_w = 3\ 125$.

For the first four values P' and P are both tensile or both compressive, and in the last three values P' is the reverse of P. If P be tension the computed values of S are to be used at once in formula (1) for finding the cross-sections, but if P be compression the length of the piece should be taken into account by formula (10) if necessary.

As a first example, let it be required to find the proper cross-section of a wrought iron bar which is to be subjected to a repeated tension ranging from 30 000 to 90 000 pounds. Here $\frac{P'}{P} = \frac{1}{3}$, and,

$$S_w = 6\,250\left(1 + \tfrac{17}{60} + \tfrac{7}{180}\right) = 8\,250.$$

Then the cross-section of the bar is $\frac{90\,000}{8\,250} = 11$ square inches.

For a second example, let it be required to find the cross-section of a wrought iron bar which is to be subjected to repeated stress ranging from 30 000 pounds compression to 90 000 tension. Here $\frac{P'}{P} = -\frac{1}{3}$, and from the formula $S_w = 4\,720$. Then the cross-section should be $\frac{90\,000}{4\,720} = 19$ square inches.

As a third example, the cross-section of a wrought iron bar is required when the stress ranges from 30 000 pounds compression to 90 000 compression. Here as before $S_w = 8\,250$. This value is now to be placed for S_c in Gordon's formula, and the cross-section may then be found as in Art. 61, for any given length.

The quantity f which is the unit-stress required to produce rupture after an enormous number of repetitions in alternating stress of equal amplitudes, was called the 'vibration strength' by Wöhler. Its value for wrought iron is about one-half and for steel a little greater than one-half the elastic limit. For cast iron u and e are greater in tension than in compression and this should be borne in mind when using formula (15).

Prob. 132. A steel bar one inch in diameter is subject to repeated stress ranging between 15 000 pounds tension and 40 000 pounds tension. Will it break after an enormous number of repetitions?

Prob. 133. Find the proper cross-section for a cast iron bar 12 feet long when subjected to repeated tension ranging from 30 000 to 90 000 pounds. Also its cross-section when subjected to repeated compression ranging between the same limits.

ART. 83. THE RESILIENCE OF MATERIALS.

When an applied stress causes a deformation or strain work is done. Thus if a tensile stress P be applied by increments to a bar, so that the stress gradually increases from 0 to the value P, the work done is the product of the average stress by the total elongation λ. This product is termed the resilience of the bar. If the stress does not exceed the elastic limit of the material the average stress is $\frac{1}{2}P$, and the work or resilience is $\frac{1}{2}P\lambda$. If the cross-section of the bar be A and its length l, the unit-stress is $\frac{P}{A}$ or S, and the unit-strain is $\frac{\lambda}{l}$ or s, so that the work done on each unit of length of the bar per unit of cross-section is $\frac{1}{2}Ss$. From formula (2) the value of s is $\frac{S}{E}$, and accordingly this work may be written,

$$K = \frac{1}{2}\frac{S^2}{E}. \qquad (16)$$

If S be the unit-stress at the elastic limit, the quantity K is called the modulus of resilience of the material.

Resilience is a measure of the capacity of a material to withstand shock, for if a shock or sudden stress be produced by a falling body, its intensity depends upon the weight and the height through which it has fallen, that is, upon its kinetic energy or work. The higher the resilience of a material the greater is its capacity to resist shocks. The modulus of resilience is a measure of this capacity within the elastic limit only.

The following are values of the modulus of resilience as computed from (16) by the use of the average constants given in Art. 5.

For timber,	$K = 3.0$ inch-pounds,
For cast iron,	$K = 1.2$ inch-pounds,
For wrought iron,	$K = 12.5$ inch-pounds,
For steel,	$K = 26.5$ inch-pounds.

The ultimate resilience of materials cannot be expressed by a rational formula, because the law of increase of elongation beyond the elastic limit is unknown. In Fig. 1 the ultimate resilience is indicated by the area between any curve and the axis of abscissas, since that area has the same value as the total work performed in producing rupture. For timber and cast iron the ratio of these areas is about the same as that of the values of K, but for wrought iron and steel the areas are nearly equal.

Prob. 134. What horse-power engine is required to strain 125 times per minute a bar of wrought iron 2 inches in diameter and 18 feet long, from 0 up to one-half its elastic limit?

Art. 84. Tables of Constants.

The following tables recapitulate the mean values of the constants of the strength of materials which have been given in the preceding pages. It is here again repeated that these values are subject to wide variations dependent on the kind and quality of the material, and for many other reasons. Timber, for instance, varies in strength according to the climate where grown, the soil, the age of the tree, the season of the year when cut, the method and duration of the process of seasoning, the part of the tree used, the knots and wind shakes, the form and size of the test specimen and the direction of its fibers, so that it is a difficult matter to state definite numerical values concerning its elasticity and strength. The quality of the material causes a yet wider variation, so wide in fact that in some cases testing machines alone could scarcely distinguish between wrought iron and steel; for while the higher grades of steel have much greater strength than the tables give, the mild structural and merchant steels may have values almost as low as the average constants for wrought iron. In general, therefore, the following values should not be used in actual cases of investigation and design except for approximate computations.

Detailed tables giving the results of experiments upon numerous kinds and qualities of materials may be found in the following books.

Wood's Resistance of Materials; New York, 1880.
Burr's Elasticity and Strength of Materials; New York, 1883.
Thurston's Materials of Engineering; New York, 1884.
Trautwine's Engineers' Pocket Book; New York, 1885.
Lanza's Applied Mechanics; New York, 1885.

TABLE I.

Material.	Mean Weight.		Coefficient of Linear Expansion.	
	Pounds per cubic foot.	Kilograms per cubic meter.	For 1° Fah.	For 1° Cent.
Timber	40	600	0.0000020	0.0000036
Brick	125	2 000	0.0000050	0.0000090
Stone	160	2 560	0.0000050	0.0000090
Cast Iron	450	7 200	0.0000062	0.0000112
Wrought Iron	480	7 700	0.0000067	0.0000121
Steel	490	7 800	0.0000065	0.0000117

TABLE II.

Material.	Elastic Limit.		Coefficient of Elasticity.	
	Pounds per square inch.	Kilograms per square centimeter.	Pounds per square inch.	Kilograms per square centimeter.
Timber	3 000	210	1 500 000	105 000
Cast Iron	6 000	420	15 000 000	1 050 000
Wrought Iron	25 000	1 750	25 000 000	1 750 000
Steel	40 000	2 800	30 000 000	2 100 000

Table III.

Material.	Ultimate Tensile Strength.		Ultimate Comparative Strength.	
	Pounds per square inch.	Kilograms per square centimeter.	Pounds per square inch.	Kilograms per square inch.
Timber	10 000	700	8 000	560
Brick	200	14	2 500	175
Stone			6 000	420
Cast Iron	20 000	1 400	90 000	6 300
Wrought Iron	55 000	3 850	55 000	3 850
Steel	100 000	7 000	150 000	10 500

Table IV.

Material.	Ultimate Shearing Strength.		Modulus of Rupture.	
	Pounds per square inch.	Kilograms per square centimeter.	Pounds per square inch.	Kilograms per square centimeter.
Timber	{ 600 3 000 }	{ 42 210 }	9 000	630
Stone			2 000	140
Cast Iron	20 000	1 400	35 000	2 450
Wrought Iron	50 000	3 500	55 000	3 850
Steel	70 000	4 900	120 000	8 400

9

$p\,x\,b\,d\,x$

$\int p\,x^2\,b\,dx = M$

$f = v\,p \quad \frac{f}{v} = p \quad \frac{f\,I}{v} = M$

$\frac{L_1}{l} \times \frac{P}{A} = E \quad \frac{P}{A} = E\,\frac{l}{L}$

$v\,d\phi = l$

$d\phi \quad \frac{v\,d\phi\,dx}{dx}\,E = P$

$E \int \frac{v^2\,d\,d\phi}{dx} = M = \frac{E\,d\,d\,I}{dx}$

$\phi = \frac{dy}{dx} \quad d\phi = \frac{d^2y}{dx} \quad \frac{d\phi}{dx} = \frac{d^2y}{dx^2}$

$M = E\,I\,\frac{d^2y}{dx^2}$

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